PAPER - 8

REAL ANALYSIS I

Objectives

To understand various limiting behavior of sequences and series

To explore the various limiting processes viz.continuity, uniform continuity, differentiability and integrability and to enhance the mathematical maturity and to work comfortably with concepts.

UNIT-I: Functions & Sequences

Functions – real valued functions – equivalence – countability and real numbers – least upper bound – definition of sequence and subsequence – limit of a sequence – convergent sequence
Ch. 1.4 to 1.7, 2.1 to 2.3 of Goldberg.

UNIT-I: Sequences [Contd...]

Divergent sequences – Bounded sequences – Monotone sequence – Operations on convergent sequences – Operations on divergent sequences – Limit superior and Limit inferior – Cauchy sequences
Ch. 2.4 to 2.10 of Goldberg.

UNIT-III: Series of Real Numbers

Convergence and Divergence – Series with non negative terms – Alternating series – conditional convergence and Absolute convergence – Test for Absolute convergence.
Ch. 3.1 to 3.4 and 3.6 of Goldberg.

UNIT-IV: Series of Real Numbers [Contd...]

Test for Absolute convergence – The class ℓ^2 – Limit of a function on the real line – Metric spaces – Limits in Metric spaces.
Ch. 3.7, 3.10, 4.1 to 4.3 of Goldberg.

UNIT-V: Continuous Functions on Metric Spaces

Functions Continuous at a point on the real line – Reformulation – Functions Continuous on a Metric Spaces – Open Sets – Closed Sets.
Ch. 5.1 to 5.5 of Goldberg

UNIT - I

SETS AND SEQUENCES

INTRODUCTION

Definition : 1

The objects in a set are called its elements or points.

Defn : 2

If b is an element of a set A, we write $b \in A$.
If b is not an element of a set A, we write $b \notin A$.

Defn : 3

If A and B are sets, then $A \cup B$ is the set of of elements in either A or B. we can write.

$$A \cup B = \{ x / x \in A \text{ or } x \in B \}$$

Defn : 4

If A and B are sets, then $A \cap B$ is the set of all elements in both A and B. we can write,

$$A \cap B = \{ x / x \in A \text{ and } x \in B \}$$

Defn : 5

we define the Empty sets φ as the set which has no elements.

Defn : 6

If A and B are sets, then $B-A$ is the set of all elements of B which are not elements of A, we write

$$A-B = \{ x / x \in B, x \notin A \}$$

Defn: 7

If every element of the Set A is an element of the Set B, we write $A \subset B$ or $B \supset A$. If $A \subset B$, we say that A is a SubSet of B.

A proper subSet of B is a SubSet $A \subset B$, such that $A \neq B$.

Defn: 8

we say that two Sets are equal, If they contain precisely, the Same elements.

NOTE:

If $A = B$ if and only if $A \subset B$ and $B \subset A$

1> Set of Natural Numbers

$$I = \{1, 2, \cdots\} = N$$

2> Set of Integers

$$Z = \{\ldots\ldots -2, -1, 0, 1, 2, \ldots\ldots\}$$

3> Set of Rational Numbers

$$Q = \left\{\frac{p}{q}, p, q \in Z, q \neq 0\right\}$$

4> Set of Irrational Numbers

Eg: $\sqrt{3}, \sqrt{5}$

5> Set of Real Numbers

$R = (-\infty, \infty)$ [both rational and irrational numbers]

6> open Interval

$$(0, 1) = \{x \in \mathbb{R}, 0 < x < 1\}$$

Eg : $a < b$, $a, b \in R$

$(a, b) = \{a, b \in R, a < x < b\}$

→ closed Interval

$[0, 1] = \{x \in R, 0 \leq x \leq 1\}$

Eg : $a < b$, $a, b \in R$

$[a, b] = \{x \in R, a \leq x \leq b\}$

Theorem :

If A, B are subsets of S, then (i) $(A \cup B)' = A' \cap B'$
and (ii) $(A \cap B)' = A' \cup B'$

proof.

i) If $x \in (A \cup B)'$ then

$x \notin A \cup B$ [Thus x is an element of neither A nor B]

$\Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \in A' \cap B'$

Hence $(A \cup B)' \subset A' \cap B' \longrightarrow (1)$

conversely.

If $y \in A' \cap B'$ then

$y \in A'$ and $y \in B'$

$\Rightarrow y \notin A$ or $y \notin B$

$\Rightarrow y \in (A \cup B)'$

Hence $A' \cap B' \subset (A \cup B)' \longrightarrow (2)$

From (1) & (2)

$(A \cup B)' = A' \cap B'$

ii) $(A\cup B)' = A' \cap B'$

Replace A by A', B by B'

$$(A'\cup B')' = A\cap B$$

Taking complement on both sides.

$$[(A'\cup B')']' = (A\cap B)'$$

$$\therefore A'\cup B' = (A\cap B)'$$

FUNCTIONS

Defn:

If A, B be two non-empty sets, A relation f from A to B is said to be a function.

If f corresponds every element of A to a unique element of B.

we write

$$\boxed{f : A \longrightarrow B}$$

A is called domain of the function.

B is called co-domain of the function

$f(A)$ is the range of the function f.

Defn: cartesion product of a set

If A, B are sets, then the cartesian product of A and B (denoted $A\times B$) is the set of all ordered pairs (a, b) where $a\in A$ and $b\in B$.

Defn: Image of x under f:

Let A and B be any two Set a function f from A into B is a SubSet of $A \times B$ (Is a Set of ordered pairs (a,b)) with the property that each $a \in A$ belongs to precisely one pair (a,b). we can write $\boxed{y = f(x)}$ Then y is called the image of x under f.

The Set A is called the domain of f.

The range of f is the Set $\{ b \in B \mid b \in f(a)$ for Some $a \}$

Defn: Inverse image

If $C \subset B$ then $f^{-1}(C)$ is defined as $\{ a \in A \mid f(a) \in C \}$ the Set of all points in the domain of f whose image are in C. If C has only one point in it, Say $C = \{y\}$.

we write $f^{-1}(y)$. The Set $f^{-1}(C)$ is called The inverse image of C under f.

Theorem:

If $f : A \longrightarrow B$ and if $X \subset B$, $Y \subset B$ Then $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$

(or) P.T The inverse image of the union of two Sets is the union of the inverse image.

proof.

Suppose $a \in f^{-1}(X \cup Y)$

$\Rightarrow f(a) \in X \cup Y$

$\Rightarrow f(a) \in X$ (or) $f(a) \in Y$

$\Rightarrow a \in f^{-1}(x)$ (or) $a \in f^{-1}(y)$

$\therefore a \in f^{-1}(x) \cup f^{-1}(y)$

Thus $f^{-1}(x \cup y) \subset f^{-1}(x) \cup f^{-1}(y)$ —— ①

conversely,

If $b \in f^{-1}(x) \cup f^{-1}(y)$

$\Rightarrow b \in f^{-1}(x)$ (or) $b \in f^{-1}(y)$

$\Rightarrow f(b) \in (x)$ (or) $f(b) \in y$

$\therefore f(b) \in x \cup y$

(ie) $b \in f^{-1}(x \cup y)$

Thus $f^{-1}(x) \cup f^{-1}(y) \subset f^{-1}(x \cup y)$ —— ②

from ① & ②

$f^{-1}(x \cup y) = f^{-1}(x) \cup f^{-1}(y)$.

Theorem :

If $f : A \to B$ and $x \subset B$, $y \subset B$ then $f^{-1}(x \cap y) = f^{-1}(x) \cap f^{-1}(y)$ (or)

P.T the inverse image of the intersection of two sets is the intersection of the inverse image.

proof.

Suppose $a \in f^{-1}(x \cap y)$

$\Rightarrow f(a) \in x \cap y$

$\Rightarrow f(a) \in x$ and $f(a) \in y$

$\Rightarrow a \in f^{-1}(x)$ and $a \in f^{-1}(y)$

$\therefore a \in f^{-1}(x) \cap f^{-1}(y)$

thus $f^{-1}(x \cap y) \subset f^{-1}(x) \cap f^{-1}(y)$ —— ①

conversely, if $b \in f^{-1}(x) \cap f^{-1}(y)$

$\Rightarrow b \in f^{-1}(x)$ and $b \in f^{-1}(y)$

$\Rightarrow f(b) \in x$ and $f(b) \in y$

$\Rightarrow f(b) \in x \cap y$

$\Rightarrow b \in f^{-1}(x \cap y)$

Thus $f^{-1}(x) \cap f^{-1}(y) \subset f^{-1}(x \cap y)$ —— ②

From ① & ②

$f^{-1}(x \cap y) = f^{-1}(x) \cap f^{-1}(y)$.

Theorem:

If $f : A \longrightarrow B$ and $X \subset A$, $Y \subset A$, Then $f(x \cup y) = f(x) \cup f(y)$. (or) P.T the image of the union of 2 set is the union of the images.

proof

Let $x \in f(x \cup y)$

$\Rightarrow f'(x) \in x \cup y$

$\Rightarrow f'(x) \in x$ or $f'(x) \in y$

$\Rightarrow x \in f(x)$ or $x \in f(y)$

$\Rightarrow x \in f(x) \cup f(y)$

$f(x \cup y) \subset f(x) \cup f(y) \longrightarrow$ ①

conversely,

Let $y \in f(x) \cup f(y)$

$\Rightarrow y \in f(x)$ or $y \in f(y)$

$\Rightarrow f^{-1}(y) \in x$ or $f^{-1}(y) \in y$

$\Rightarrow f^{-1}(y) \in x \cup y$

$\Rightarrow y \in f(x \cup y)$

i.e., $f(x) \cup f(y) \subset f(x \cup y) \longrightarrow (2)$

from (1) & (2)

$f(x \cup y) = f(x) \cup f(y)$

NOTE :

$f(x \cap y) = f(x) \cap f(y)$ for $x \subset A$, $y \subset A$

Defn :-

composition of functions

If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ then we define the function gof by $gof(x) = g[f(x)]$ $(x \in A)$ (i.e.) the image of x under gof defined to be the image of $f(x)$ under g, the fun gof is called composition of f with g.

Defn :

If $f : A \longrightarrow B$ and $g : A \longrightarrow B$ we define $f+g$ as the function whose value at $x \in A$ is equal to $f(x) + g(x)$.

i.e.) $(f+g)(x) = f(x) + g(x)$ $x \in A$

In notation

$f+g = \{x, f(x) + g(x)\}$, $x \in A$

It is clear that

$f+g : A \longrightarrow R$

Similarly, $f-g$ and fg defined by

$$(f-g)x = f(x) - g(x),\ x \in A$$

$$fg(x) = f(x)\,g(x),\ x \in A$$

If $g(x) \neq 0$ for all

$x \in A$, we define f/g

$$(f/g)x = \frac{f(x)}{g(x)},\ x \in A$$

Defn: If $f : A \longrightarrow R$ and c is a real number $(c \in R)$, the function cf is defined by

$$(cf)(x) = c\,[f(x)],\ x \in A$$

Defn:

If $f : A \longrightarrow R$, $g : A \longrightarrow R$ Then $\max(f,g)$ is the function defined by $\max(f,g)(x) = \max[f(x), g(x)]$, $x \in A$

and $\min(f,g)(x) = \min[f(x), g(x)]$, $x \in A$

Defn:

If $f : A \rightarrow R$, then $|f|$ is the function defined by $|f|x = |f(x)|$ $(x \in A)$

NOTE:

If a, b are real Numbers then

$$\max(a,b) = \frac{|a-b| + a + b}{2}$$

$$\min(a,b) = \frac{-|a-b| + a + b}{2}$$

Defn: Characteristic function of A

If $A \subset S$, Then χ_A is defined as

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \in A' \end{cases}$$

properties of characteristic function of A.

i) If A, B are subsets of S.

$$\chi_{A \cup B} = \max(\chi_A, \chi_B)$$

ii) $\chi_{A \cap B} = \min(\chi_A, \chi_B)$

$= \chi_A \chi_B$

iii) $\chi_{A-B} = \chi_A - \chi_B$ provided $B \subset A$

iv) $\chi_{A'} = 1 - \chi_A$

v) $\chi_S = 1$

vi) $\chi_\phi = 0$

proof for the properties

i) If A, B are subsets of S.

$$\chi_{A \cup B} = \max(\chi_A, \chi_B)$$

proof

case i)

Let $x \in A \cup B$

$\Rightarrow \chi_{A \cup B}(x) = 1$ —— (1)

Let $x \in A \cup B$

$\Rightarrow x \in A$ or $x \in B$

$\chi_A(x) = 1$ or $\chi_B(x) = 1$

$$\max(\chi_A, \chi_B)(x) = \max\{\chi_A(x), \chi_B(x)\}$$

$$= \max(1,1)$$

$$= 1 \longrightarrow 2$$

From 1 & 2

Let $x \in A \cup B$, we

have $\chi_{A \cup B}(x) = \max(\chi_A, \chi_B)(x)$ —— I

case ii)

Let $x \notin A \cup B$ Then

$\chi_{A \cup B}(x) = 0 \longrightarrow 3$

Let $x \notin A \cup B$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow \chi_A(x) = 0$ or $\chi_B(x) = 0$

$$\max(\chi_A, \chi_B)(x) = \max(\chi_A(x), \chi_B(x))$$

$$= \max(0,0)$$

$$= 0$$

From 3 & A

Let $x \notin A \cup B$, we have

$\chi_{A \cup B}(x) = \max(\chi_A, \chi_B)(x)$ —— II

From I & II we have

$$\chi_{A \cup B} = \max(\chi_A, \chi_B)$$

ii) $\chi_{A \cap B} = \min(\chi_A, \chi_B) = \chi_A \chi_B$

proof,

case i) let $x \in A \cap B$

$\Rightarrow \chi_{A \cap B}(x) = 1$ and also $\longrightarrow$ 1

$x \in A \cap B$

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow \chi_A(x) = 1$ and $\chi_B(x) = 1$

$\Rightarrow \min(\chi_A, \chi_B)(x) = \min(\chi_A(x), \chi_B(x))$

$= \min(1, 1)$

$= 1 \longrightarrow$ 2

$\Rightarrow (\chi_A, \chi_B)(x) = (\chi_A(x) ; \chi_B(x))$

$= (1, 1)$

$= 1 \longrightarrow$ 3

From 1, 2, 3 let $x \in A \cap B$ we have

$\chi_{(A \cap B)}(x) = \min(\chi_A, \chi_B)(x)$

$= \chi_A(x) \cdot \chi_B(x) \longrightarrow$ I

case(ii)

let $x \notin A \cap B$ then

$\chi_{A \cap B} = 0 \longrightarrow$ 4

Let $x \notin A \cap B$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow \chi_A(x) = 0$ and $\chi_B(x) = 0$

$$\min \chi_{A\cap B}(x) = \min(\chi_A(x), \chi_B(x))$$

$$= \min(0,0)$$

$$= 0 \text{ ——— } (5)$$

$$(\chi_A, \chi_B)(x) = (\chi_A(x), \chi_B(x))$$

$$= (0,0)$$

$$= 0 \text{ ——— } (6)$$

From (4), (5), (6)

Let $x \notin A \cap B$, we have

$$\chi_{A\cap B}(x) = \min(\chi_A, \chi_B)(x)$$

$$= \chi_A(x), \chi_B(x) \text{ ——— } II$$

From I & II we have

$$\chi_{A\cap B} = \min(\chi_A, \chi_B) = \chi_A \chi_B$$

(iii) $\chi_{A-B} = \chi_A - \chi_B$ (provided $B \subset A$)

proof: case (i)

Let $x \in A-B$ then

$\chi_{A-B}(x) = 1$ ——— (1)

and $x \in A-B$

$\Rightarrow x \in A$ and $x \notin B$

$\chi_A(x)=1$ and $\chi_B(x)=0$

consider,

$$(\chi_A-\chi_B)(x)=\chi_A(x)-\chi_B(x)$$
$$=1-0$$
$$=1 \longrightarrow (2)$$

From (1) & (2)

let $x\in A-B$, we have

$$\chi_{(A-B)}(x) = (\chi_A-\chi_B)(x) \longrightarrow I$$

case (ii)

let $x\notin A-B$

$$\Rightarrow \chi_{A-B}(x)=0 \longrightarrow (3)$$

Also $x\notin A-B$

$\Rightarrow x\notin A$ and $x\notin B$

$\Rightarrow \chi_A(x)=0$ and $\chi_B(x)=0$

In both case

$$(\chi_A-\chi_B)(x)=\chi_A(x)-\chi_B(x)$$
$$=0 \longrightarrow (4)$$

From (3) & (4)

$$\chi_{(A-B)}(x) = (\chi_A-\chi_B)(x) \longrightarrow II$$

from I, II

$$\chi_{(A-B)} = \chi_A-\chi_B$$

iv) $\chi_{A'} = 1 - \chi_A$

proof: case (i)

let $x \in A'$ then

$\chi_{A'} = 1 \longrightarrow ①$

also $x \in A' \Rightarrow x \notin A$

$\Rightarrow \chi_A(x) = 0$

consider.

$(1-\chi_A)(x) = 1 - \chi_A(x)$

$= 1 - 0$

$= 1 \longrightarrow ②$

from ① + ②

let $x \in A'$, we have

$\chi_{A'} = 1 - \chi_A$

case ii)

let $x \notin A'$ then

$\chi_{A'}(x) = 0 \longrightarrow ③$

and also

$x \notin A' \Rightarrow x \in A$

$\chi_A(x) = 1 \longrightarrow ④$

consider.

$(1-\chi_A)(x) = 1 - \chi_A(x)$

$= 1 - 1$

$= 0$

from ③ + ④

let $x \notin A'$, we have

$\chi_{A'}(x) = 1 - \chi_A(x)$ (or) $\chi_{A'} = 1 - \chi_A$

(v) $\chi_S = 1$

proof:

Since, $\forall x \in S$

we have

$\chi_S(x) = 1, \forall x$

$\chi_S = 1$

(vi) $\chi_\phi = 0$

proof. $\chi_\phi = \chi_{S'}$ $\qquad [\because \chi_{A'} = 1 - \chi_A]$

$= 1 - \chi_S$

$= 1 - 1$

$= 0$

(or)

$S - S = \phi$

$\chi_{S-S} = \chi_S - \chi_S \qquad \because \{\chi_{A-B} = \chi_A - \chi_B\}$

$\chi_\phi = 1 - 1$

$= 0$

$\chi_\phi = 0$

Equivalence Countability:

Defn:

If $f: A \rightarrow B$, then f is called one-to-one (or) (1-1),

If $f(a_1) = f(a_2)$, $(a_1, a_2) \in A$

NOTE:

If f is 1-1 and $b = f(a_1)$ then $b \neq f(a_2)$ for any $a_2 \in A$ distinct from a_1.

Example :

1. f is defined by $f(x) = x^2$, $(-\infty < x < \infty)$ is not 1-1, function.

2. f is defined by $f(x) = x^2$, $(0 \le x \le \infty)$ is 1-1.

Defn :

If $f : A \longrightarrow B$ and f is 1-1 then the function f^{-1} (called the inverse function of f) is defined as follows If $f(a) = b$, then $f^{-1}(b) = a$ (b is the range of f)

Defn :

If $f : A \Longrightarrow B$ and f is 1-1, then f is called a 1-1 correspondence (b/w A and B) If there exists a 1-1 correspondence b/w the sets A and B, then A and B are called equivalent.

NOTE :

1) Every set A is equivalent to itself

2) If A and B are equivalent then B and A are equivalent.

3) If A and B are equivalent and B and C are equivalent, then A and C are equivalent.

Defn : [an infinite set]

The set A is said to be infinite if, for each positive integer n, A contains a subset with precisely n element.

Eg :

1) The set of all positive integers

$I = \{1, 2, \cdots\}$, I is an infinite set.

2) The Set of all real Numbers R is also an infinite Set.

Defn :

The Set A is Said to be Countable, if A is equivalent to be the Set I of positive integers.

i.e., $A \sim I$

An uncountable Set is a infinite Set which is not countable, eg : The Set of all real number.

<u>Composition of function :</u>

Eg :

If $f(x) = 1+\sin x$, $-\infty < x < \infty$

$g(x) = x^2$, $0 < x < \infty$

$(g \circ f)(x) = g[f(x)]$

$= g[1+\sin x]$

$= (1+\sin x)^2$ $\qquad \because g(x) = x^2$

$(g \circ f)x = 1+\sin^2 x + 2\sin x \quad (-\infty < x < x)$

problem

2) Let $f(x) = 2x$, $-\infty < x < \infty$ can you think of fun g and h which Satisfies the two equations, $g \circ f = 2gh$ and $h \circ f = h^2 - g^2$

Soln//

we can take $g(x) = \sin x$, and $h(x) = \cos x$

$(g \circ f)(x) = g[f(x)]$

$= g[2x]$

$= \sin 2x$

$= 2\sin x \cos x$

$(g o f)(x) = 2gh$

$(h o f)(x) = h[f(x)]$

$= h[2x]$

$= \cos 2x$

$= \cos^2 x - \sin^2 x$

$= h^2(x) - g^2(x)$

$h o f = h^2 - g^2$

NOTE :

1) If f is one to one, if $(a_1 \neq a_2)$

$\Rightarrow f(a_1) \neq f(a_2)$

2) one to one mapping is called injection.

Defn : on to function

If $f : A \longrightarrow B$, if range of f is all of B, ie $f(A) = B$, we say that f is a function from A onto B.

Defn: A set S is called finite and is said to contain n elements of $S \sim \{1, 2, \cdots n\}$

The integer n is called the cardinal number of S.

NOTE :

i) ϕ is a finite set with cardinal number 0.

ii) If A is countable $\Rightarrow f : I \longrightarrow A$ is 1-1 $\Rightarrow$ $\{f(1), f(2), \cdots\} = A$, we write $f(i) = a_i$ then $A = \{a_1, a_2, a_3, \cdots\}$ (ie) a set A is countable, if its elements are arranged with labels $1, 2, \cdots$

problems :

1. Let $A = \{1/2, 2/3, 3/4, \cdots\}$ Then A is countable.

Soln

Since the mapping $f : I \longrightarrow A$

$$f(x) = \left\{ \frac{n}{n+1}, \forall n \in I \right\}$$

is bijective. Hence A is countable.

2) P.T the set of all integers is countable.

Soln

$$Z = \{ \cdots -2, -1, 0, 1, 2 \cdots \}$$

Let $f : I \longrightarrow Z$

$$f(x) = \begin{cases} \frac{n-1}{2}, & n = 1, 3, 5 \cdots\cdots \\ -n/2, & n = 2, 4 \cdots \end{cases}$$

Clearly f is bijective map, then the set of integers are countable.

* when the mapping is bijection it is countable.

Theorem :

Statement :

If $A_1, A_2, A_3, \cdots\cdots$ are countable sets, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

(or)

The countable union of countable sets is countable.

proof.

Let $A_1, A_2, \ldots$ are countable

we write,

$$A_1 = \{a_{11}, a_{12}, a_{13}, \ldots$$

$$A_2 = \{a_{21}, a_{22}, a_{23}, \ldots$$

$$A_3 = \{a_{31}, a_{32}, a_{33}, \ldots$$

$$\vdots$$

$$A_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots$$

Here a_{ij} is the j^{th} element of A_i

Define the height of a_{ij} as $i+j$, then a_{11} is the only element of height 2 a_{12} and a_{21} are the only element of height and so.

i.e. for any positive integer $m \geq 2$, there are only $(m-1)$ elements of height m.

$\therefore$ we arrange (count) the elements of $\bigcup_{n=1}^{\infty} A_n$, according to their height as $a_{11}, a_{21}, a_{12}, a_{22}, a_{31}, \ldots$

being careful to remove any a_{ij} that had already been countable. pictorially the element of $\bigcup_{n=1}^{\infty} A_n$ can be listed as following array.

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdot \\ a_{41} & a_{42} & & & \\ \vdots & & & & \end{matrix}$$

This Scheme actually counts every a_{ij}

$\therefore \bigcup_{n=1}^{\infty} A_n$ is countable corollary.

The Set of all Rational number is countable

proof,

The Set of all rational number is the union as

$Q = \bigcup_{n=1}^{\infty} E_n$ where E_n is the Set of rational which can be written with denomination n.

$$\text{ie) } E_1 = \left\{ \cdots \frac{-2}{1}, \frac{-1}{1}, 0, \frac{1}{1}, \frac{2}{1}, \cdots \right\}$$

$$E_2 = \left\{ \cdots \frac{-2}{2}, \frac{-1}{2}, 0, \frac{1}{2}, \frac{2}{2} \cdots \right\}$$

$$\vdots$$

$$E_n = \left\{ \cdots -2/n, -1/n, 0, 1/n, 2/n \cdots \right\}$$

now, each E_n is clearly equivalent to the Set of all integers and is thus countable by the theorem $\bigcup_{n=1}^{\infty} E_n$ is countable.

Hence the Set of all rational is the countable union of countable Set,

$\therefore$ The Set of all rational number is countable.

Theorem:

Statement:

If B is an infinite subset of the countable set A, then B is countable.

(or)

A subset of countable set is countable.

Proof:-

Let A be any countable set and let $B \subset A$.

If B is finite, then there is nothing to prove.

If B is infinite then A is countably infinite. Then we can write A as an infinite sequence.

Let $A = \{a_1, a_2, a_3, \ldots\}$

Then each element of b is an a_i,

Let n_1 be the least positive integer such that $a_{n_1} \in B$, let n_2 be the next smallest positive integer greater than n_1 such that $a_{n_2} \in B$ and so on.

Then it is evident that the mapping $f: I \longrightarrow B$, $f(K) = a_{nK}$ is bijective, B is countable.

Problem:-

The set of all rational number in $[0,1]$ is countable.

Sol,

Q is the set of all rational numbers.

Let A be the set of all rational numbers in $[0,1]$

A is an infinite subset and Q is countable

$\therefore$ A is countable [$\because$ by the previous theorem]

Q. P.T any infinite set contain a countable subset.

Proof.

Let A be an infinite set $A \neq \phi$

We can choose $x_1 \in A$

$A - \{x_1\} \neq \phi$

We can choose $x_2 \in A - \{x_1\}$

Such that $x_1 \neq x_2$. Proceeding in this way, we can choose $x_n \in A$, such that

$x_1 \neq x_2 \neq x_3 \neq \ldots\ldots \neq x_{n-1} \neq x_n$

$\therefore A - \{x_1, x_2, \ldots\ldots x_n\} \neq \phi$

and so on, then

$\{x_1, x_2, \ldots\ldots x_n\}$ is a subset of B also

$\{x_1, x_2, x_3 \ldots\ldots x_n)$ is countable.

3. S.T the set of all ordered pair of integer is countable (ie) [$I \times I$ is countable]

Sol.

$f: I \times I \longrightarrow Q$ defined by

$f(\langle m,n \rangle) = 2^m . 2^n \quad (m,n \in Z)$

To prove that f is 1-1

$f(\langle m,n \rangle) = f(\langle p,q \rangle)$

$2^m . 2^n = 2^p . 2^q$

$\frac{2^m . 2^n}{2^p . 2^q} = 1$

$(2^{m-p} . 2^{n-q}) = 1^0$

$\Longrightarrow m-p=0, \; n-q=0$

$\Longrightarrow m=p, \; n=q$

$\langle m,n \rangle = \langle p,q \rangle$

$\therefore f$ is 1-1.

$\therefore I \times I \sim f(I \times I)$

But $f(I \times I) \subset Q$

$\therefore f(I \times I)$ is countable [$\because Q$ is countable

$\Longrightarrow I \times I$ is countable.

4) If A and B are countable, S.T $A \times B$ is countable

Sol.

$A = \{a_1, a_2, \ldots\}, B = \{b_1, b_2, \ldots\}$

$A \times B = \{ \langle a_n, b_m \rangle \; n,m \in I \}$

Define,

$f: A \times B \longrightarrow I$ defined $f(\langle m,n \rangle) = 2^m . 3^n$

To prove : f is 1-1

$$f(\langle m,n\rangle) = f(\langle p,q\rangle)$$

$$2^m \cdot 3^n = 2^p \cdot 3^q$$

$$2^{m-p}\, 3^{n-q} = 1$$

$$\Rightarrow m-p=0 \;,\; n-q=0$$

$$m=p \;,\; n=q$$

$$\therefore \langle m,n\rangle = \langle p,q\rangle$$

$\therefore f$ is 1-1

$A\times B \sim f(A\times B)$, $f(A\times B) \subset I$,

$\therefore f(A\times B)$ is countable . [$\because I$ is countable]

$\Rightarrow A\times B$ is countable

P.T The set of all polynomial function with integer co-efficient is countable .

proof.

$$P_n = f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, \quad a_k \in \mathbb{Z}$$

$(k = 0,1,2\ldots n)$

The set of all n^{th} degree polynomial with integer co-efficient , with each .

$$f(x) = a_0 + a_1x + \cdots + a_nx^n \in P_n$$

associate $(a_0, a_1, \ldots\ldots a_n) \in \mathbb{Z}^{n+1}$

$\therefore P_n \sim \mathbb{Z}^{n+1}$, but $\mathbb{Z}^{n+1}$ is countable

$\therefore P_n$ is countable $(n = 0,1,\ldots)$

Let P is denote the set of all polynomial with integer co-efficient

$$P = \bigcup_{n=1}^{\infty} P_n \quad [\because \text{countable union of countable set is countable}]$$

$\therefore P$ is countable.

REAL NUMBERS

Every real number x can be written in decimal expansion

$$x = b \cdot a_1 a_2 \cdots$$

$$= b + \frac{a_1}{10} + \frac{a_2}{10} + \cdots$$

where a_i's are integers such that $0 \le a_i \le 9$

This expansion is unique except for cases such as $x = \frac{1}{2}$ which can expanded in two different forms.

$$\frac{1}{2} = 0.5000 \text{ and}$$

$$\frac{1}{2} = 0.4999$$

Every number $x \in [0,1]$ can be expanded conversely. $x = 0 \cdot a_1 a_2 a_3 \cdots$

we assume that every decimal is of the form $b \cdot a_1 a_2 a_3 \cdots$ is the decimal expansion same real numbers.

Theorem :

The set $[0,1] = \{x ; 0 \le x \le 1\}$ is uncountable

proof

suppose $[0,1]$ where countable

then $[0,1] = \{x_1, x_2 \cdots\}$ where every number is $[0,1]$ occurs among the x_i expanding each x_i in decimal we have,

$$x_1 = 0 \cdot a_{11}\, a_{12} \cdots$$
$$x_2 = 0 \cdot a_{21}\, a_{22} \cdots$$
$$\vdots$$
$$x_n = 0 \cdot a_{n1}\, a_{n2} \cdots a_{nn}$$

let b_1 be any integer from 0 to 8 such that $b_1 \neq a_{11}$. Then let b_2 be any integer from 0 to 8 such that $b_2 \neq a_{22}$

In general for each $n = 1, 2 \cdots$ let b_n be any integer from 0 to 8 such that $b_n \neq a_{nn}$.

let $y = 0 \cdot b_1 b_2 \cdots b_n \cdots$

Then for any n, the decimal expansion for y differs from the decimal expansion for x_n. Since $b_n \neq a_{nn}$ moreover, the decimal expansion for y is unique, Since no b_n is equal to 9.

Hence $y \neq x_n$ for every n and $0 \leq y \leq 1$ which contradicts the assumption that every number in $[0,1]$ occurs among the x_i.

$\therefore [0,1]$ is uncountable.

Corallary :

The Set of all real numbers are uncountable.

proof :-

Suppose R is countable. Its infinite subset $[0,1]$ is also countable. [$\because$ The infinite subset of countable set is countable]

which is a contradition [$\because$ $[0,1]$ is uncountable]

$\therefore$ R is uncountable.

Cantor Set K :

The Cantor Set K is the Set of all ternary number in $[0,1]$ which have a ternary expansion without digit 1.

Eg :

$1/3 = 0.033$

$\frac{1}{3} = 0.666\ldots$

are in K but any number x such that

$1/3 < x < 2/3$

$\therefore$ are not in K

Theorem :

The Cantor Set is uncountable

proof.

Defn $f : K \longrightarrow [0,1]$ by $f(x) = y$

$x = 0.b_1, b_2 \ldots \; b_i = 0$ (or) 2

$y = 0.a_1, a_2 \ldots \ldots a_i = \frac{b_i}{2}$

clearly f is onto

$\therefore$ $f : K \longrightarrow [0,1]$ is clearly. It is easy to show that f is onto but $[0,1]$ is uncountable

$\therefore$ K is uncountable

Theorem :

P.T if B is countable subset of the uncountable set A. then A-B is uncountable.

proof: Given A is uncountable, B is countable and $B \subset A$.

To prove that A-B is uncountable

$$A = (A-B) \cup B$$

is countable. [∵ countable union of countable set is countable]

Which is contradiction to the fact that, A-B is uncountable.

(ii) P.T The set of all irrational are uncountable.

proof, Q = The set of all rational number is countable.

R = The set of all real numbers is uncountable.

A = The set of all irrational number is uncountable.

To prove A is uncountable

Suppose A is countable, Then $Q \cup A$ is countable

∴ R is countable

which is a contradiction

∴ A is uncountable.

Least upper bound :

Defn :

The subset $A \subset R$ is said to be bounded above if there is an number $N \in R$ such that $x \leq N$ for every $x \in A$, N is called upper bound.

Lower bound:

The SubSet $A \subset R$ is said to be bounded below, if there exists a number $M \in R$ such that $M \leq x$ for every $x \in A$, M is called lower bounded.

Bounded Set :

the SubSet $A \subset R$ is bounded below and bounded above then [we Say that] A is bounded.

Eg :

1) $I = \{1, 2, \cdots\}$ is bounded above and below,

2) $[0, 1]$ is bounded

3) $(0, 1)$ is bounded

4) The Set all real numbers less than or equal to 1 (≤ 1)

ie, $\{x : -\infty < x \leq 1\}$ is bounded below,

5) $E = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$, 1 is an upper bound and 0 is lower bounded for E. $\therefore E$ is bounded.

Defn:

least upper bound (lub) :-

Let the SubSet S of R bounded above the number L is called the least upper bound for S, if

i) L is an upper bound for S.

ii) NO number smaller than L is an upper bound for S.

Similarly l is called the greatest lower bound (glb) for the Set S bounded below If l is a lower bound for S and no number greater than l is a lower bound for S.

Defn: Least upper bound Axiom :-

If A is any non-empty subset of R that is bounded above, then A has least upper bounded in R.

Theorem :

If A is any non-empty subset of R that is bounded below, then A has a greatest lower bound in R.

proof:

Let $B \subset R$ be the set of all $x \in R$. such that $-x \in A$

If M is a lower bound for A, then $-M$ is an upper bound for B, for if $x \in B$, then $-x \in A$ and so

$M \leq -x$, $x \leq -M$

Hence B is bounded above by least upper bound axiom, B has a least upper bound.

Sequences of real numbers

Defn:

A Sequence $S = \{s_i\}_{i=1}^{\infty}$ of real numbers is a function from I (the set of +ve Integer) in to R (the set of real number)

$$\text{ie; } S : I \longrightarrow R$$

$$\text{Here } S(i) = s_i \ (i = 1, 2 \cdots]$$

Sequence $S = \{s_i\}_{i=1}^{\infty}$ can also be written as $s_1, s_2 \cdots$ The number s_i ($i = 1, 2 \cdots$) is called the i^{th} term of the Sequence $\{s_1, s_2 \cdots\}$ is called the range of the Sequence.

Sub Sequences :

Defn :

If $S = \{S_n\}_{n=1}^{\infty}$ is a Sequence of real number and $N = \{n_i\}_{i=1}^{\infty}$ is a subsequence of the sequence of the integers then the composite function $S \circ N$ is called a Subsequence of S.

NOTE :

i) for $i \in I$; $N(i) = n_i$

$$S \circ N(i) = S[N(i)]$$
$$= S(n_i)$$
$$= S_{n_i}$$

and hence $S \circ N = \{S_{n_i}\}_{i=1}^{\infty}$

ii) The SubSequence of $S_1, S_2, \ldots$ is $S_{n_1}, S_{n_2}, \ldots$

Example :

1. Consider the Sequence $S(i) = \frac{1}{i}$, $N(i) = 2^i$ ($i = 1, 2 \ldots$

$$S(i) = \frac{1}{i}$$
$$N(i) = 2^i \ (i = 1, 2 \ldots$$

$$S \circ N(i) = S[N(i)]$$
$$= S(2^i) \ (i = 1, 2 \ldots$$
$$= \frac{1}{2^i} \ (i = 1, 2, \ldots$$

$\therefore$ SubSequence of $1, \frac{1}{2}, \ldots$ is $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3} \ldots$

2) If the Sequence 1, 0, 1, 0, ... is denoted by B and we defined $N = \{n_i\}_{i=1}^{\infty}$ by $n_i = (2i-1)$. Find the Sequence.

$$B \circ N(i) = B[(2i-1)]$$

$$= b_{2i-1}$$

$$B \circ N = b_1, b_3 \dots$$

$$= 1, 1, 1, \dots$$

3) If $c = \{c_n\}_{n=1}^{\infty} = \{\sqrt{n}\}_{n=1}^{\infty}$ and $N = n_i = \{i^4\}$ then find the SubSequence.

$$C \circ N(i) = c[n_i]$$

$$= c(i^4)$$

$$= \sqrt{i^4}$$

$$C \circ N(i) = i^2$$

Fibonacci number :-

Let $\{s_n\}_{n=1}^{\infty}$ be the Sequence defined by

$s_1 = 1$, $\boxed{s_{n+1} = s_n + s_{n-1}}$ where $n = 2, 3 \dots$

The numbers in s_n are called Fibonacci number.

4) Let $\{s_n\}_{n=1}^{\infty}$ be the Sequence defined by $s_1 = 1$, $s_2 = 1$, $s_{n+1} = s_n + s_{n-1}$, Find Sequence up to s_8.

i.e., $s_{n+1} = s_n + s_{n-1}$

$n=2$
$$S_3 = S_2 + S_1$$
$$S_3 = 1+1$$
$$S_3 = 2$$

$n=3$
$$S_4 = S_3 + S_2$$
$$= 2+1$$
$$S_4 = 3$$

$n=4$
$$S_5 = S_4 + S_3 = 3+2 = 5$$

$n=5$
$$S_6 = S_5 + S_4 = 5+3 = 8$$

$n=6$ $$S_7 = S_6 + S_5 = 8+5 = 13$$

$n=7$ $$S_8 = S_7 + S_6 = 13+8 = 21$$

2) write the formula for S_n

a) $2, 1, 4, 3, 6, 5 \ldots$
$$S_n = \begin{cases} n+1, & n = 1, 3, 5 \ldots \\ n-1, & n = 2, 4, 6 \ldots \end{cases}$$

b) $1, 0, 1, 0, \ldots$
$$S_n = \begin{cases} 1, & \text{if } n = 1, 3, 5, \ldots \\ 0, & \text{if } n = 2, 4, 6, \ldots \end{cases}$$

c) $1, 3, 6, 10, 15 \ldots$
$$S_n = S_{n-1} + n, \quad n = 2, 3, \ldots$$

and $S_1 = 1^2$

d) $1, -4, 9, -16,$ $$S_n = (-1)^{n+1} n^2,$$

(2) 1, 1, 1, 2, 1, 3 . . . .

$$S_n = \begin{cases} 1, \text{ if } n = 1, 3 \dots \\ n/2, \text{ if } n = 2, 4 \dots \end{cases}$$

If $S = \{S_n\}_{n=1}^{\infty} = \{2n-1\}_{n=1}^{\infty}$ and $N = n(i) = \{i^2\}_{i=1}^{\infty}$

Find S_5, S_9, S_{n_2}, S_{n_3}. Is N is a subsequence of $k\}_{i=1}^{\infty}$

Sol:

$S_n = 2n-1$

$n = 5$

$\therefore S_5 = 2(5)-1 = 9$ //

$n = 9$

$\therefore S_9 = 2(9) - 1 = 17$ //

$n = 2 \quad = N(2) = 2^2 = 4$ //

Limit of a Sequence.

let $\{S_n\}_{n=1}^{\infty}$ be a sequence of real numbers

[we say that S_n approaches the limit L (as n approaches ∞) if for every $\epsilon > 0$ there is a positive integer N such that.

$$|S_n - L| < \epsilon \quad n \geq N$$

we write $\lim\limits_{n \to \infty} S_n = L$ (or) $S_n \longrightarrow L$ $(n \longrightarrow \infty)$

1. using the definition (Limit of sequence)

$\lim\limits_{n \to \infty} \frac{1}{n} = 0 \quad [n = 1, 2, 3 \dots]$

Sol:

let $\epsilon > 0$ be given to find the positive integer $N \in I$ such that $|S_n - L| < \epsilon$, $n \geq N$

i.e., $|1/n - 0| < \epsilon$, $n \geq N$

$|1/n| < \epsilon$, $n \geq N$

$1/n < \epsilon$, choose $N > 1/\epsilon$

2) S.T the Sequence $\{s_n\}_{n=1}^{\infty}$ where $s_n = 1$, $(n = 1, 2 \ldots)$ has the limit $L = 1$.

Soln

Let $L = 1$

$\epsilon > 0$ be given. To find $N \in I$ such that

$|s_n - L| < \epsilon$, $n \geq N$

i.e., $|1 - 1| < \epsilon$, $n \geq N$; $\epsilon > 0$, $n \geq N$

choose $N = 1$

$\therefore \lim_{n \to \infty} s_n = 1$,

3) P.T the Sequence $\{s_n\}_{n=1}^{\infty}$ where $s_n = n$ $(n = 1, 2, \ldots)$ does not have the limit.

Soln

Let $\lim_{n \to \infty} s_n = L$, where $s_n = n$ $(n = 1, 2 \ldots)$

Given $\epsilon = 1$, there exists a positive integer $N \in I$ such that

$|s_n - L| < 1$, $n \geq N$

i.e., $|n - L| < 1$, $n \geq N$

$-1 < n - L < 1$, $(n \geq N)$

$L - 1 < n < L + 1$, $(n \geq N)$

All integer Value of $n \geq N$ lies between $L - 1$ and $L + 1$. This impossible.

$\therefore \{s_n\}$ has no limit.

N) S.T The Sequence $\{s_n\}$ where $s_n = (-1)^n$ $(n = 1, 2 \ldots$ has no limit.

Sol.

suppose $\lim_{n \to \infty} s_n = L$

where $s_n = (-1)^n$, $n = (1, 2, \ldots$

Given $\epsilon = 1/2$, $\exists$ $N \in I$

such that $|s_n - L| < 1/2$

i.e, $|(-1)^n - L| < 1/2$, $n \geq N$

If n is even, $|1 - L| < 1/2$, $n \geq N$

If n is odd, $|-1 - L| < 1/2$, $n \geq N$

consider, $2 = |2|$

$= |1+1|$

$= |1 + L + 1 - L|$

$\leq |1+L| + |1-L|$

$< 1/2 + 1/2$

$2 < 1 \Rightarrow \epsilon$

$\therefore \{(-1)^n\}$ has no limit.

1) Let $s_n = \dfrac{2n}{n + 4n^{1/2}}$, $n = 1, 2 \ldots$ P.T $\lim_{n \to \infty} s_n = 2$, using the definition of limit.

Sol.

Given $\epsilon > 0$, $\exists$ $N \in I$ such that $|s_n - L| < \epsilon$, $n \geq N$

i.e, $L = 2$

$$\left| \frac{2n}{n + 4n^{1/2}} \right| < \epsilon \, , \; n \geq N$$

$$\left|\frac{2n-2n-8n^{1/2}}{n+4n^{1/2}}\right| < \epsilon, \quad n \geq N$$

$$\left|\frac{8n^{1/2}}{n+4n^{1/2}}\right| < \epsilon, \quad n \geq N$$

$$\left|\frac{8n^{1/2}}{n+4n^{1/2}}\right| < \epsilon, \quad n \geq N$$

$$\frac{8}{n^{1/2}+4} < \epsilon, \quad n \geq N$$

$$8/n^{1/2} < n^{1/2}, \quad n \geq N$$

Squaring on both sides

$$\Rightarrow \frac{64}{\epsilon^2} < n, \quad n \geq N$$

$$\text{i.e., } n > \frac{64}{\epsilon^2}, \quad n \geq N$$

$$\text{Choose } N > \frac{64}{N^2}$$

$$\therefore \lim_{n\to\infty} S_n = 2.$$

Another method :

using the definition of limit :

$$\lim_{n\to\infty} \frac{2n}{n+4n^{1/2}} = \lim_{n\to\infty} \cancel{n}\left[\frac{2}{1+\frac{4}{n^{1/2}}}\right]$$

$$= \frac{2}{1+\lim_{n\to\infty}\frac{4}{n^{1/2}}}$$

$$= \frac{2}{1+0} = 2.$$

Theorem : 1

(6) If $\{S_n\}_{n=1}^{\infty}$ is a Sequence of non-negative numbers and if $\lim_{n\to\infty} S_n = L$ then $L \geq 0$.

proof :
on the contrary assume that $L < 0$

Let $\epsilon = -\frac{L}{2}$, $\lim_{n\to\infty} S_n = L$, then there exists $N \in I$ such that $|S_n - L| < \epsilon$, $n \geq N$

$$|S_n - L| < -L/2 , n \geq N$$

i.e., $\frac{L}{2} < S_n - L < -L/2$, $n \geq N$

$$3L/2 < S_n < L/2 , n \geq N$$

$$S_n < L/2 , n \geq N$$

$$\Rightarrow \Leftarrow \text{ [Iff] }$$

$$\therefore L \geq 0 \quad [\because S_n \geq 0]$$

2. If $\{S_n\}$ is a Sequence of real numbers if $S_n \leq M$, $(n \in I)$ and $\lim_{n\to\infty} S_n = L$. prove $L \leq M$.

proof
$$S_n \leq M$$
$$M - S_n \geq 0$$
$$\lim_{n\to\infty} (M - S_n) \geq 0$$
$$M - \lim_{n\to\infty} S_n \geq 1$$
$$M - L \geq 0$$
$$M \geq L$$
$$\Rightarrow L \leq M$$

A) prove $\lim_{n\to\infty} \frac{2n}{n+3} = 2$

Sol, Let $\epsilon > 0$, be given to find $N \in I$ such that

$$\left|\frac{2n}{n+3} - 2\right| < \epsilon, \; n \geq N$$

$$\left|\frac{2n - 2n - 6}{n+3}\right| < \epsilon, \; n \geq N$$

$$\frac{6}{n+3} < \epsilon, \; n \geq N$$

$$6/n < \epsilon, \; n \geq N$$

$$n > \frac{6}{\epsilon}, \; n \geq N$$

choose $N > \frac{6}{\epsilon}$, that $\lim_{n\to\infty} \frac{2n}{n+3}$

$$\Rightarrow \lim_{n\to\infty} \frac{2n}{n(1+3/n)}$$

$$\Rightarrow \lim_{n\to\infty} \frac{2}{1+3/n}$$

$$\Rightarrow \frac{2}{1+\lim_{n\to\infty} 3/n} = 2$$

convergent Sequence :

If the Sequence of the real numbers $\{s_n\}_{n=1}^{\infty}$ has the limit L, we say that

$\{s_n\}_{n=1}^{\infty}$ converges to L.

If $\{s_n\}_{n=1}^{\infty}$ does not have a limit, we say that $\{s_n\}_{n=1}^{\infty}$ diverges

Eg :

1. $1, 1/2, 1/3, \ldots$ converges to 0.

2. $1, 1, 1, \ldots$ converges to 1.

3. $1, -1, 1, -1, \ldots$ diverges.

4. $1, 2, 3, \ldots$ - diverges.

Example :

Find $N \in I$ such that $\frac{1}{\sqrt{n+1}} < 0.03$ $(n \geq N)$

Soln

$$\frac{1}{\sqrt{n+1}} < 0.03 \quad (n \geq N)$$

$$\frac{1}{0.03} < \sqrt{n+1}, \; n \geq N$$

Squaring on both sides

$$\frac{1}{0.0009} < n+1, \; n \geq N$$

$$n+1 > 1111 \tfrac{1}{9}, \; n \geq N$$

$$n > 1110 \cdot \tfrac{1}{9}, \; n \geq N$$

choose $N = 1111$

2. P.T $\lim_{n \to \infty} \frac{1}{\sqrt{n+1}} = 0$

Soln let $\epsilon > 0$ be given

To find $N \in I$ such that $|S_n - L| < \epsilon, n \geq N$

ie) $\left| \frac{1}{\sqrt{n+1}} - 0 \right| < \epsilon \quad , n \geq N$

$$\frac{1}{\sqrt{n+1}} < \epsilon, \quad n \geq N$$

$$\frac{1}{n+1} < \varepsilon^2, \; n \geq N$$

$$1/\varepsilon^2 < n+1, \; n \geq N$$

$$n+1 > 1/\varepsilon^2, \; n \geq N$$

$$n > 1/\varepsilon^2 - 1, \; n \geq N$$

$$N = 1/\varepsilon^2$$

Theorem 2 :

If the Sequence of real numbers $\{S_n\}_{n=1}^{\infty}$ is convergent to L, then $\{S_n\}_{n=1}^{\infty}$ cannot converges to a limit distinct from L. that is $\lim\limits_{n\to\infty} S_n = L$ and $\lim\limits_{n\to\infty} S_n = M$ then $L = M$.

proof :

Given $\lim\limits_{n\to\infty} S_n = L$

$\lim\limits_{n\to\infty} S_n = M$

To prove that $L = M$

$\lim\limits_{n\to\infty} S_n = L$, $\lim\limits_{n\to\infty} S_n = M$

Given $\varepsilon > 0$ there exists $N_1 \in I$, such that

$$|S_n - L| < \varepsilon/2 \quad (n \geq N_1) \longrightarrow (1)$$

and $\varepsilon > 0$, $\exists\, N_2 \in I$, such that

$$|S_n - M| < \varepsilon/2, \; (n \geq N_2) \longrightarrow (2)$$

Take $N = \max(N_1, N_2) \longrightarrow (3)$

consider, $|L - M| = |L - S_n + S_n - M|$

$$\leq |L - S_n| + |S_n - M| \quad \{\because \text{using } (1)\,(2) \;\&\; (3)\}$$

$< \varepsilon/_2 + \varepsilon/_2 \quad (n \geq N)$

$|L-M| < \varepsilon$

Since ε is arbitrary

$|L-M| = 0 \Rightarrow L = M$

Theorem 3 :

If the Sequence of real numbers $\{S_n\}_{n=1}^{\infty}$ is convergent to L, then any Sub Sequence of $\{S_n\}_{n=1}^{\infty}$ is also convergent to L.

proof ;

Given $\lim\limits_{n\to\infty} S_n = L$, $\varepsilon > 0$, $\exists$ $N \in I$ such that

$|S_n - L| < \varepsilon$, $n \geq N$

consider the subsequence $\{S_{n_k}\}_{k=1}^{\infty}$

choose a +ve integer K, such that $n_k > N$, $k \geq K$.

Then

$|S_{n_k} - L| < \varepsilon$, $(k \geq K)$

$\lim\limits_{n\to\infty} S_{n_k} = L$

Corollary .

All Subsequence of a convergent Sequence of the real numbers convergent to the Same limit .

proof .

By using the theorem ② & ③ .

Let $\{S_n\}_{n=1}^{\infty}$ be a Sequence converges to L. Then by the theorem ①, $\{S_n\}_{n=1}^{\infty}$ converges to no other Limit

By the theorem ③ all the subsequence $\{S_n\}_{n=1}^{\infty}$ converges to L .

Example :

i) A Sequence which diverges but has convergent Subsequence

$$S_n = (-1)^n, \ n \in I$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (-1)^n$$

$$= (+1) (or) (-1)$$

$\therefore (-1)^n$ diverges

The Subsequence 1, 1, 1, ... converges for to 1, (ie) $\{(-1)^n\}_{n=1}^{\infty}$ converges to 1 and −1, −1 ... converges to − 1 (ie) $\{(-1)^n\}_{n=1}^{\infty}$ converges to −1.

ii) If a Sequence which diverges and has only divergent Subsequence

$$\text{Let } S_n = n, \ n \in I$$

$\lim_{n\to\infty} S_n = \infty$ and all its Subsequence also divergens.

Exercise :

$a, b \in R$ S.T $||a| - |b|| \leq |a-b|$ then P.T $\{|S_n|\}_{n=1}^{\infty}$ converges to $|L|$ if $\{S_n\}_{n=1}^{\infty}$ converges to L.

proof :

$$|a| = |a-b+b|$$

$$\leq |a-b| + |b|$$

$$\Rightarrow |a| - |b| \leq |a-b| \longrightarrow (1)$$

Interchanging 'a' and 'b'

we get $|b| - |a| \leq |b-a|$

$\therefore [|a|-|b|] \leq |a-b| \longrightarrow ②$

From ① & ②

$\Rightarrow |a|-|b| \leq |a-b|,\ a,b \in R$

Given $\lim_{n \to \infty} S_n = L$, given $\epsilon > 0$

$\exists\ N \in I$ such that

$|S_n - L| < \epsilon,\ n \geq N$

$||a|-|b|| \leq |a-b|,\ a,b \in R$

$||S_n|-|L|| \leq |S_n - L|$

$< \epsilon$

$\therefore \lim_{n \to \infty} |S_n| = |L|$ //

NOTE :

Does the converse of above results true

need not be true.

Give an example of a sequence $\{S_n\}_{n=1}^{\infty}$ of real numbers for which $\{|S_n|\}_{n=1}^{\infty}$ converges but $\{S_n\}_{n=1}^{\infty}$ does not converges.

Sol.

Let $S_n = (-1)^n,\ n \in I$

$|S_n| = |(-1)^n| = 1,\ n \in I$

$\lim_{n \to \infty} |S_n| = 1,$ but $\lim_{n \to \infty} S_n = \lim_{n \to \infty} (-1)^n$

$|S_n|$ converges to 1.

$= 1(a) - 1$

$\{S_n\}_{n=1}^{\infty}$ diverges

P.T If $\{|S_n|\}_{n=1}^{\infty}$ converges to 0, then $\{S_n\}_{n=1}^{\infty}$ converges to 0.

Sol/

Then $\lim_{n\to\infty} |s_n| = 0$

Given $\epsilon > 0$, $\exists$ $N \in I$

such that

$|s_n - l|$ Here $s_n = |s_n|$, $l = 0$

$||s_n| - 0| < \epsilon$, $n \geq N$

$|s_n| < \epsilon$, $n \geq N$

$|s_n - 0| < \epsilon$, $n \geq N$

$\lim_{n\to\infty} s_n = 0$ //

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UNIT - II

DIVERGENT SEQUENCES

Defn [Diverges to ∞]

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers, we say that s_n approaches (∞ infinity as n approaches infinity if for any real number $M>0$, there is a the integer N such that $s_n \geq M$, $(n \geq N)$

i.e. we write $\lim_{n\to\infty} s_n = \infty$ [diverges to $+\infty$]

Eg: $\{n\}_{n=1}^{\infty}$ diverges to ∞

For given $M>0$, choose $N \in I$

such that $n > M$ then

$s_n \geq n > M$, $n \geq N$

Definition (Diverges to $-\infty$)

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers, we say that s_n approaches $-\infty$ as n approaches infinite, if for any real numbers $M > 0$, there is a positive integer such that $S_n < -M$, $(n \geq N)$

(ie) we write it $\lim_{n \to \infty} S_n = -\infty$

Eg: $\{-n\}_{n=1}^{\infty}$ diverges to $-\infty$

NOTE: The Sequence $1, -2, 3, -4, \ldots$ does not approaches either $+\infty$ or $-\infty$.

But its subsequences $1, 3, 5 \ldots$ diverges to ∞ and has subsequence $-2, -4, \ldots$ which diverges to $-\infty$

RESULT: If the sequence $\{S_n\}_{n=1}^{\infty}$ diverges to ∞, then so does any subsequences of $\{S_n\}_{n=1}^{\infty}$

Oscillating Sequence:

Defn: If the Sequence $\{S_n\}_{n=1}^{\infty}$ of real numbers diverges but not diverges to infinity $(+\infty)$ and does not diverge to minus infinity $(-\infty)$ we say that $\{S_n\}_{n=1}^{\infty}$ is oscillates.

Example:

1. $\{(-1)^n\}_{n=1}^{\infty}$
2. $1, 2, 1, 3, 1, 4, \ldots$

Exercise:

Lable each of the following Sequence

1) $\{e^{1/n}\}_{n=1}^{\infty}$

Soln

$S_n = e^{1/n}$

$n=1 \quad s_1 = e^1$

$n=2 \quad s_2 = e^{1/2}$

$\vdots$

$\lim_{n\to\infty} \quad s_\infty = e^{1/\infty} = e^0 = 1$

$\therefore$ the Sequence $\{e^{1/n}\}_{n=1}^{\infty}$ is converges to 1.

2) $\{e^n\}_{n=1}^{\infty}$

$n = 1 \quad s_1 = e^1 = e$

$n = 2 \quad s_2 = e^2 = e^2$

$\vdots \quad \vdots$

$\lim_{n\to\infty} \quad s_n = \lim_{n\to\infty} e^{\infty} = \infty$

$\therefore$ The Sequence $\{e^n\}_{n=1}^{\infty}$ is diverges to ∞.

3) $\{\sin n\pi\}_{n=1}^{\infty}$

let $S_n = \sin n\pi$

$s_1 = \sin\pi = 0$

$s_2 = \sin 2\pi = 0 \; \cdots = 0$

$\vdots$

$\lim_{n\to\infty} \sin n\pi = 0$

$\therefore$ The Sequence converges to 0.

4) $\{-n^2\}_{n=1}^{\infty}$

$s_n = -n^2$

$s_1 = -(1)^2 = -1$

$\vdots$

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} (-1)^2 = -\infty$

$\therefore$ It diverges to $-\infty$

5) $\{\sin n\pi/_2\}_{n=1}^{\infty}$

$S_n = \sin n\pi/_2$

$S_1 = \sin \pi/_2 = 1$

$S_2 = \sin 2\pi/_2 = \sin \pi = 0$

$S_3 = \sin 3\pi/_2 =$

$\vdots$

$\lt_{n\to\infty} S_n = \lt_{n\to\infty} \sin n\pi$

$= \infty$

$\therefore$ It is an oscillating sequence.

prove that $\{\sqrt{n}\}$ diverges to infinity

proof For a given $M > 0$, to prove there exists $N \in I$

such that $S_n > M \ (n \geq N)$

$\Rightarrow \sqrt{n} > M \ (n \geq N)$

$\Rightarrow n > M^2 \ (n \geq N)$

choose $N > M^2$, then $S_n > M \ (n \geq N)$

$\therefore \lt_{n\to\infty} \sqrt{n} = \infty$

prove that $\{\sqrt{n+1} - \sqrt{n}\}_{n=1}^{\infty}$ is convergent

sol. let $S_n = \sqrt{n+1} - \sqrt{n}$

$$= \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} - \sqrt{n} \times \sqrt{n+1} + \sqrt{n}}$$

$$= \frac{n+1-n}{\sqrt{n+1} - \sqrt{n}}$$

$$= \frac{1}{\sqrt{n}\sqrt{1+1/n}+1}$$

$$= \lim_{n\to\infty} \left[\frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{1+1/n}} \right]$$

$= 0$

NOTE :

suppose $\{s_n\}_{n=1}^{\infty}$ converges to zero then $\{(-1)^n s_n\}_{n=1}^{\infty}$

Soln

Given $\lim_{n\to\infty} s_n = 0$

$\Rightarrow$ given $\epsilon > 0$ $\exists$ $N \in I$ such that

$|s_n - 0| < \epsilon \quad (n \geq N)$

consider,

$|(-1)^n s_n - 0| < \epsilon \quad (n \geq N)$

ie,

$$|(-1)^n s_n - 0| = |(-1)^n s_n|$$
$$= |s_n|$$
$$= |s_n - 0|, \; n \geq N$$
$$< \epsilon$$

$\therefore$ $\{(-1)^n s_n\}_{n=1}^{\infty}$ converges to zero

P.T : The sequence of real number $\{s_n\}_{n=1}^{\infty}$ diverges to ∞ then $\{-s_n\}_{n=1}^{\infty}$ diverges to $-\infty$

Sol.

Given $\lim_{n\to\infty} s_n = \infty$ for every $\epsilon > 0$, $\exists$ $N \in I$ such that.

$s_n > M \quad (n \geq N)$

$-s_n > -M \quad (n \geq N)$

$\lim_{n\to\infty} -s_n = -\infty$

NOTE : 1

Suppose $\{ S_n \}_{n=1}^{\infty}$ converges to $L \neq 0$, $\{(-1)^n S_n\}_{n=1}^{\infty}$ oscillates.

NOTE : 2

suppose $\{ S_n \}_{n=1}^{\infty}$ diverges to ∞, then $\{(-1)^n S_n\}_{n=1}^{\infty}$ is oscillates.

BOUNDED SEQUENCES :

Defn :

Let $\{S_n\}_{n=1}^{\infty}$ be a Sequence of real numbers (ie) $\{S_n\}_{n=1}^{\infty}$ is a function from I into R. The range of the Sequence $\{ S_1, S_2, \ldots \}$ is a subset of R.

The Sequence is said to be bounded above if its range is bounded above.

The Sequence is said to be bounded below if its range is bounded below,

NOTE :

1) If the Sequence diverges to ∞, then the Sequence is not bounded

2) An oscillating Sequence, may or may not be bounded

Defn

A Sequence is bounded if it is bounded above and bounded below (ie) $\{ S_n \}_{n=1}^{\infty}$ is bounded if and only if $\exists\ M > 0$ such that $|S_n| \leq M$ $(n \in I)$

Theorem :

If the Sequence of real numbers $\{S_n\}_{n=1}^{\infty}$ is convergent, then $\{S_n\}_{n=1}^{\infty}$ is bounded.

proof:

Given $\{S_n\}_{n=1}^{\infty}$ converges,

Let $\lim_{n\to\infty} S_n = L$ given $\epsilon = 1$ there exists $N \in I$

such that

$$|S_n - L| < 1 \quad (n \geq N)$$

consider $|S_n| = |S_n - L + L|$

$$\leq |S_n - L| + |L|, \quad (n \geq N)$$

$$< 1 + |L| \longrightarrow (1) \quad (n \geq N)$$

Let $M = \max(|S_1|, |S_2|, \ldots |S_{n-1}|) \longrightarrow (2)$

from (1) & (2)

$$|S_n| < 1 + |L| + M \quad (n \in I)$$

$\therefore \{S_n\}_{n=1}^{\infty}$ is bounded.

NOTE :

1) Say true or false

If the Sequence +ve number is not bounded then the Sequence diverges to ∞.

Ans : True

2) Given an example of the Sequence $\{S_n\}_{n=1}^{\infty}$ which is not bounded but for which $\lim_{n\to\infty} \frac{S_n}{n} = 0$.

eg : Let $S_n = \sqrt{n}$, $n = 1, 2, 3 \ldots$

$\{\sqrt{n}\}_{n=1}^{\infty}$ is not bounded.

$$\lim_{n\to\infty} \frac{s_n}{n} = \lim_{n\to\infty} \frac{\sqrt{n}}{n}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}}$$

$$= 0 \,//$$

Monotone Sequence :

Defn :

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

If $s_1 \le s_2 \le s_3 \cdots \le s_n \le s_{n+1} \le \cdots$ then $\{s_n\}_{n=1}^{\infty}$ is called non-decreasing.

If $s_1 \ge s_2 \ge s_3 \ge \cdots \ge s_n \ge s_{n+1} \ge \cdots$

Then $\{s_n\}_{n=1}^{\infty}$ is called non-increasing.

A monotone sequence is a sequence which is either non-decreasing (or) non increasing [for both].

Theorem :

A non-decreasing sequence which is bounded above is convergent.

Proof :

Let $\{s_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of real numbers which is bounded above

Let $A = \{s_1, s_2 \cdots\} \subset R$

A - the range of $\{s_n\}_{n=1}^{\infty}$

A is bounded above

By least upper bound axiom A has a least upper bound.

Let least upper bound $(A) = M$ — ①

To prove that

$$\lim_{n\to\infty} s_n = M$$

given $\varepsilon > 0$ by ① $M-\varepsilon$ is not an least upper bound function A. there exists $N \in I$ such that $S_n > M-\varepsilon (n \geq N)$

One the otherhand, Since M is an upper bound for A.

② $\Rightarrow M \geq S_n, n \in I$

$\Rightarrow S_n < M+\varepsilon, n \in I$ ——③

② + ③

$M-\varepsilon < S_n < M+\varepsilon, (n \geq N)$

$-\varepsilon < S_n - M < \varepsilon, (n \geq N)$

$|S_n - M| < \varepsilon, (n \geq N)$

$\therefore \lim_{n \to \infty} S_n = M$ //

Corollary,

The Sequence $\{(1+1/n)^n\}_{n=1}^{\infty}$ is convergent.

proof,

let $S_n = (1+1/n)^n$

By using Binomial Theorem

$$S_n = 1 + n \cdot 1/n + \frac{n(n-1)}{2!} \cdot 1/n^2 + \cdots + \frac{n(n-1)}{1,2\cdots n} \cdots 1/n^n$$

for $K = 1, 2, \cdots n$ the $(K+1)^{th}$ term on the right is

$$\frac{n(n-1)(n-K+1)}{1,2\cdots K} \cdot \frac{1}{n^K}$$

Which equals,

$$\frac{1}{1,2\cdots K}(1-1/n)(1-2/n)\cdots\left(1-\frac{K-1}{n}\right) \longrightarrow ①$$

If we expand S_{n+1} obtain $(n+2)$ terms 1 are more than for S_n

and for $k = 1, 2, \cdots n$ the $(k+1)^{th}$ term is

$$\frac{1}{1,2\cdots k}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right)$$

Which is greater than or equal to the quantity ① this

Shows that $S_n \leq S_{n+1}$, $n \in I$

i.e. $\{S_n\}_{n=1}^{\infty}$ is non-decreasing but also.

$$S_n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \cdots$$

$$\cdots + \frac{n(n-1)(n(n-1)}{1,2,\cdots n} \cdot \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1(1-1/n)}{1-2} + \frac{1(1-1/n)(1-2/n)}{1,2,3} + \cdots + \frac{1(1-1/n)}{1,2\cdots n}$$

$$< 1 + 1 + \frac{1}{1,2} + \frac{1}{1,2,3} + \cdots + \frac{1}{1,2\cdots n}$$

$$< 1 + \left\{1 + 1/2 + 1/2^2 + 1/2^3 + \cdots + 1/2^{n-1}\right\}$$

$$< 1 + \frac{(1-1/2)^n}{1-1/2} \quad \left[\because S_n = a + ar + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}\right]$$

$$< 1 + (1-1/2)^n / 1-1/2$$

$$< 1 + 2$$

$$< 3$$

$\therefore \{S_n\}_{n=1}^{\infty}$ is bounded above.

$\therefore$ By the above theorem

$\{S_n\}$ is convergent. It is customary to

denote

$$\lim_{n\to\infty} S_n = e,$$

$\therefore \lim_{n\to\infty} (1+1/n)^n = 0$

e lies b/w 2 and 3 $(2 < e < 3)$

$S_1 = 1+1 = 2$

$S_2 = (1+1/2)^2 > 2$

$\vdots$

$S_n < 3 \ (n \in I)$

$\therefore 2 < S_n < 3$

$2 < \lim_{n\to\infty} S_n \leq 3$

$2 < e \leq 3$

Theorem :

a non - decreasing Sequence which is not bounded above diverges to infinity.

proof

let $\{S_n\}_{n=1}^{\infty}$ be a non-decreasing Sequence which is not bounded above.

$\{S_n\}_{n=1}^{\infty}$ is not bounded above

for every $M > 0$ $\exists$ $N \in J$ such that $S_N > M$

$\{S_n\}$ is a non - decreasing

$\therefore S_n > M \ (\forall n \geq N)$

$\lim_{n\to\infty} S_n = \infty$

Theorem :

A non_increasing sequence which is not bounded below diverges to $-\infty$

proof.

let $\{S_n\}_{n=1}^{\infty}$ be a non-increasing Sequence

Which is not bounded below .

$\{S_n\}_{n=1}^{\infty}$ is not bounded below ,

for every $M>0$, $\exists$ $n\in I$ such that $S_n < -M$

$\{S_n\}$ is non-increasing

$\therefore S_n < -M$, $(\forall\ n \geq N)$

$\lim\limits_{n\to\infty} S_n = -\infty$

Statement :

Which of the following Sequence are monotone

1) $\left\{\frac{1}{1+n^2}\right\}_{n=1}^{\infty}$

Sol, $\left\{\frac{1}{1+n^2}\right\}_{n=1}^{\infty} = 1/2, 1/5, 1/10, 1/17, \ldots$

It is non-increasing monotone.

2) $\{2n+(-1)^n\}$

Sol

$\{2n+(-1)^n = 1, 5, 5, 9, 9, 13,$

It is non-decreasing monotone

If $\{S_n\}^{\infty}$ is non-decreasing and bounded above and $L =$ limit S_n. P.T $S_n \leq L$ $(n\in I)$

Sol,

Given, $\{S_n\}_{n=1}^{\infty}$ is non-decreasing and bounded. above. also $L = \lim\limits_{n\to\infty} S_n$

$L = l.u.b. (A)$

where $A = \{S_1, S_2, \ldots\ldots\}$

$\implies S_n \leq L$, $(\forall\ n \in I)$

1. If $\{S_n\}_{n=1}^{\infty}$ is non-increasing and bounded below and $L = \lim_{n\to\infty} S_n$. P.T $S_n \geq L$ $(n \in I)$

Proof:-

Given $\{S_n\}^{\infty}$ is non-increasing and bounded below. Also

$$l = \lim_{n\to\infty} S_n$$

$$l = g.l.b\,(A), \text{ where } A = \{S_1, S_2, \ldots\}.$$

$$S_n \geq l \quad [n \in I]$$

2. If $S_n = \frac{10^n}{n!}$ find $N \in I$ such that $S_{n+1} < S_n$ $(n \geq N)$

Sol,

$$\text{Given } S_n = \frac{10^n}{n!}$$

$$S_{n+1} < S_n \quad (n \geq N)$$

$$\frac{10^{n+1}}{(n+1)!} < \frac{10^n}{n!} \quad (n \geq N)$$

$$10 < n+1 \quad (n \geq N)$$

$$n > 9, \quad (n \geq N).$$

Operation on convergence sequences (properties)

theorem:-

If $\{S_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequence of real number, if $\lim S_n = L$ and if $\lim t_n = M$, then $\lim_{n\to\infty} (S_n + t_n) = L + M$

(or)

The limit of the sum of the two convergence sequence is the sum of the limits.

Proof:-

Given $\lim_{n\to\infty} S_n = L$, $\lim_{n\to\infty} t_n = M$

Given $\varepsilon > 0$, there exist $N_1 \in I$.

such that

$$|S_n - L| < \frac{\varepsilon}{2} \quad (n \geq N_1) \quad \text{——— (1)}$$

and $\varepsilon > 0$. there exist $N_2 \in I$ such that

$$|t_n - M| < \varepsilon/2 \quad (n \geq N_2) \longrightarrow (2)$$

Take $N = \max(N_1, N_2)$

From (1) & (2)

$$|(s_n + t_n) - (L+M)| \leq |s_n - L| + |t_n - M|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$< \varepsilon, \quad (n \geq N)$$

$$\therefore \lim_{n\to\infty} (s_n + t_n) = (L+M)$$

Theorem :

If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers.
If $c \in R$ and if $\lim_{n\to\infty} s_n = L$, then $\lim_{n\to\infty} cs_n = cL$

Proof :

If $c=0$, the theorem is obviously

Assume $c \neq 0$, given $\lim_{n\to\infty} s_n = L$

given $\varepsilon > 0$. there exist $N \in I$, such that

$$|s_n - L| < \frac{\varepsilon}{|c|} \quad (n \geq N)$$

Consider $|c s_n - cL| = |c(s_n - L)|$

$$= |c|\,|s_n - L|$$
$$< |c| \frac{\varepsilon}{|c|} < \varepsilon, \quad (n \geq N)$$

$$\therefore \lim_{n\to\infty} cs_n = cL, \quad (n > n)$$

NOTE: A non-increasing which is bounded below is convergent. A non-increasing sequence is not bounded below diverges to ∞

Theorem :-

(a) If $0 < x < 1$, then $\{x^n\}_{n=1}$ convergies to 0.

(b) If $1 < x < \infty$. the $\{x^n\}_{n=1}^{\infty}$ divergies to ∞

proof :

(a) If $0 < x < 1$, then

$$x^{n+1} = x, x^n < x^n \mid x^{n+1} < x^n \Rightarrow S_{n+n} < S_n$$

$\therefore \{x^n\}$ is non-increasing

Since $x^n > 0$ for $n \in I$, $\{x^n\}_{n=1}^{\infty}$ is bounded below.

We know that A non-increasing sequences bounded below in convergent.

$$\text{Let } \lim_{n\to\infty} x^n = L$$

$$\text{Also } \lim_{n\to\infty} CS_n = CL$$

$$\Rightarrow \lim_{n\to\infty} x^{n+1} = \lim_{n\to\infty} x x^n = xL$$

(ie) $\{x^{n+1}\}_{n=1}^{\infty}$ converges to xL

But $\{x^{n+1}\}_{n=1}^{\infty}$ is a subsequence of $\{x^n\}$

$$\therefore \lim_{n\to\infty} x^{n+1} = L \qquad \lim_{n\to\infty} x^n = L$$

$$xL = L$$

$$(x-1) L = 0$$

$$L = 0 \quad [\because x \neq 1, 0 < x < 1]$$

$$\therefore \lim_{n\to\infty} S_n = 0$$

(b) If $x > 1$, $x^{n+1} = x \cdot x^n$ $(n \in I$

$> x^n$

$\{x^n\}$ is non-decreasing sequence

We shall show that $\{x^n\}$ is unbounded.

On the contrary assume that $\{x^n\}$ is bounded above then $\{x^n\}$ converges.

$\therefore \lim_{n \to \infty} x^n = L$ (say)

By the same argument in (a)

we have $xL = L$

$(x-1) L = 0$

$L = 0$ $(x > 1)$ which is a contradiction

$\therefore$ $\{x^n\}$ is bounded above.

$\therefore$ $\{x^n\}$ is divergesto ∞

Theorem

If $\{S_n\}_{n=1}^{\infty}$ and $\{t_n\}^{\infty}$ are sequence of real numbers. If $\lim_{n \to \infty} S_n = L$ and if $\lim_{n \to \infty} t_n = M$ then $\lim_{n \to \infty} (S_n - t_n) = L - M$.

Proof:-

We know that

$\lim_{n \to \infty} (S_n + t_n) = L + M$ and

$\lim_{n \to \infty} C S_n = CL$

$$\Rightarrow \lim_{n \to \infty} |S_n - t_n| = \lim_{n \to \infty} [S_n + (-t_n)]$$

$$= \lim_{n \to \infty} [S_n] + \lim_{n \to \infty} (-t_n)$$

$$= \lim_{n \to \infty} S_n - \lim_{n \to \infty} t_n.$$

$$= L - M.$$

Corollary:

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are convergent sequence of real number if $s_n \leq t_n$ $(n \in I)$ and if $\lim_{n\to\infty} s_n = L$, $\lim_{n\to\infty} t_n = M$ then $L \leq M$.

Proof :-

$$M - L = \lim_{n\to\infty} t_n - \lim_{n\to\infty} s_n$$

$$= \lim_{n\to\infty} (t_n - s_n)$$

Since $s_n \leq t_n$ $(n \in I)$

$$t_n - s_n \geq 0 \quad (n \in I)$$

$$\therefore \lim_{n\to\infty} (t_n - s_n) \geq 0$$

$$M - L \geq 0$$

$$M \geq L$$

$$L \leq M$$

Lemma :-

If $\{s_n\}^{\infty}$ is a sequence of real numbers which converges to L, then $\{s_n^2\}_{n=1}^{\infty}$ is converges to L^2.

Proof:- consider $|s_n^2 - L^2| = |(s_n - L)(s_n + L)|$

$$= |s_n - L|\,|s_n + L| \quad \text{--- (1)}$$

Since $\{s_n\}_{n=1}^{\infty}$ converges

$\Rightarrow \{s_n\}$ is bounded

For some $M>0$, $|s_n| \le M$, $(n \in I)$

$|s_n + L| \le |s_n| + |L|$

$\le M + |L| \longrightarrow ②$

$\lim_{n\to\infty} s_n = L$, given $\epsilon > 0$, there exists $N \in I$

Such that $|s_n - L| < \dfrac{\epsilon}{M+|L|}$ $(n \ge N)$

① $\Rightarrow |s_n^2 - L^2| < \dfrac{\epsilon}{M+|L|} \cdot M+|L|$ $(n \ge N)$
{using 2

$|s_n^2 - L^2| < \epsilon$ $(n \ge N)$

$\therefore \lim_{n\to\infty} s_n^2 = L^2$ //

Theorem :

If $\{s_n\}_{n=1}^{\infty}$ & $\{t_n\}_{n=1}^{\infty}$ are sequence of real numbers if $\lim_{n\to\infty} s_n = L$ and if $\lim_{n\to\infty} t_n = M$ then

$\lim_{n\to\infty} s_n t_n = LM$.

proof:

consider,

$|s_n t_n - LM| = |s_n t_n - L t_n + L t_n - LM|$

$\le |t_n(s_n - L)| + |L(t_n - M)|$

$\le |t_n||s_n - L| + |L||(t_n - M)| —①$

Given $\{t_n\}_{n=1}^{\infty}$ converges

$\Rightarrow \{t_n\}$ is bounded.

For some $A>0$. $|t_n| \le A$, $[n \in I] \longrightarrow ②$

$\lim_{n\to\infty} s_n = L$, $\lim_{n\to\infty} t_n = M$

Given $\epsilon > 0$ there exists $N_1 \in I$ such that

$$|s_n - L| < \frac{\epsilon}{2A} \quad (n \geq N_1) \longrightarrow (3)$$

Given $\epsilon > 0$, there exists $N_2 \in I$ such that

$$|s_n - L| < \frac{\epsilon}{2|L|} \quad (n \geq N_2 \longrightarrow (4)$$

Take $N = \max(N_1, N_2)$

using $N = \max(N_1, N_2)$

$$(1) \Rightarrow |s_n t_n - LM| < \cancel{A} \frac{\epsilon}{\cancel{2A}} + |L| \cdot \frac{\epsilon}{2|L|} \quad (n \geq N)$$

$$< \epsilon$$

$$\therefore \lim_{n \to \infty} s_n t_n = LM.$$

Lemma :

If $\{t_n\}_{n=1}^{\infty}$ is the sequence of real numbers, if $\lim_{n \to \infty} t_n = M$ where $M \neq 0$, then $\lim_{n \to \infty} \left(\frac{1}{t_n}\right) = \frac{1}{M}$.

proof.

Here $M \neq 0$

$\Rightarrow M > 0$ or $M < 0$

Let $M > 0$,

consider, $\left|\frac{1}{t_n} - \frac{1}{M}\right| = \left|\frac{M - t_n}{t_n M}\right|$

$$= \left|\frac{M - t_n}{t_n M}\right| \longrightarrow (1)$$

Given $\lim_{n \to \infty} t_n = M$, $M > 0$ there exists $N_1 \in I$ such that $|t_n - M| < \frac{M}{2}$, $(n \geq N_1)$

$$-\frac{M}{2} < t_n - M < \frac{M}{2}$$

$$\frac{M}{2} < t_n < \frac{3M}{2}$$

$$t_n > \frac{M}{2} \quad (n \geq N_1 \longrightarrow (2)$$

Also given $\varepsilon > 0$, $\exists\, N_2 \in I$ such that

$$|t_n - M| < M^2 \varepsilon/2 \quad (n \geq N_2) \quad \text{--- (3)}$$

Take $N = \max(N_1, N_2)$

using (2) & (3) in (1)

$$\left| \frac{1}{t_n} - \frac{1}{M} \right| = \frac{|M - t_n|}{|t_n M|}$$

$$< \frac{M^2 \varepsilon/2}{M \cdot M/2} \quad (n \geq N)$$

$$\left| \frac{1}{t_n} - \frac{1}{M} \right| < \varepsilon \quad (n \geq N)$$

$$\therefore \lim_{n\to\infty} \left(\frac{1}{t_n}\right) = \frac{1}{M}.$$

Lemma :

If $\{t_n\}_{n=1}^{\infty}$ is the sequence of real numbers, if $\lim_{n\to\infty} t_n = M$ where $M \neq 0$, then $\lim_{n\to\infty}\left(\frac{1}{t_n}\right) = \frac{1}{M}$

proof.

Here $M \neq 0$

$\Rightarrow M > 0$ or $M < 0$

let $M > 0$

Consider $\left| \frac{1}{t_n} - \frac{1}{M} \right| = \left| \frac{M - t_n}{t_n M} \right|$

$$= \frac{|M - t_n|}{|t_n M|} \quad \text{--- (1)}$$

Given $\lim_{n\to\infty} t_n = M$, $M > 0$, there exists $N_1 \in I$ such that $|t_n - M| < M/2$, $(n \geq N_1)$

$$-M/2 < t_n - M < M/2$$

$$M/2 < t_n < 3M/2$$

$$t_n > M/2 \quad (n \geq N_1) \quad \text{--- (2)}$$

Also given $\varepsilon > 0$, $\exists$ $N_2 \in J$ such that

$$|t_n - M| < M^2 \varepsilon/2 \quad (n \geq N_2) \text{ --- } (3)$$

Taking $N = \max(N_1, N_2)$

using (2) & (3) in (1)

$$\left|\frac{1}{t_n} - \frac{1}{M}\right| = \frac{|M - t_n|}{|t_n M|}$$

$$< \frac{M^2 \frac{\varepsilon}{2}}{M \cdot \frac{M}{2}} \quad (n \geq N)$$

$$\therefore \left|\frac{1}{t_n} - \frac{1}{M}\right| < \varepsilon \quad (n \geq N$$

$$\therefore \lim_{n\to\infty}\left(\frac{1}{t_n}\right) = \frac{1}{M}$$

If $M < 0$ is the same proof taken $(-t_n)$ instead of t_n.

Theorem :.

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequence of real numbers if $\lim_{n\to\infty} s_n = L$ and if $\lim_{n\to\infty} t_n = M$ where $M \neq 0$,

The $\lim_{n\to\infty} \frac{s_n}{t_n} = L/M$.

proof. By using the previous theorem and Lemma.

$$\lim_{n\to\infty}\left(\frac{s_n}{t_n}\right) = \lim_{n\to\infty} s_n \cdot \lim_{n\to\infty} 1/t_n$$

$$= L \cdot 1/M$$

$$= L/M$$

Exercise : problems.

1) $\lim_{n\to\infty} \dfrac{3n^2-6n}{5n^2+4n} = \dfrac{3}{5}$

Sol:

$$\lim_{n\to\infty} \frac{3n^2-6n}{5n^2+4n} = \lim_{n\to\infty} \frac{\cancel{n^2}(3-6/n)}{\cancel{n^2}(5-4/n)}$$

$$= \lim_{n\to\infty} \frac{(3-6/n)}{5-4/n}$$

$$= \frac{3-0}{5-0}$$

$$= 3/5$$

2) $\lim_{n\to\infty} \sqrt{n}\,(\sqrt{n+1}-\sqrt{n}) = 1/\sqrt{2}$

Sol.

$$S_n = \sqrt{n}\;(\sqrt{n+1}-\sqrt{n}) \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{\sqrt{n}\,(n+1-n)}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{\sqrt{n}}{\sqrt{n}}\left[\frac{1}{\sqrt{\frac{n+1}{n}}+1}\right]$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n}+1}$$

$$= \frac{1}{\sqrt{1}+1}$$

$$= \frac{1}{1+1}$$

$$= 1/2$$

prove that (i) $\lim_{n\to\infty}(1+\frac{1}{n})^{n+1} = e$ Also (ii) P.T $\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n} = e$

Sol:

i) $\lim_{n\to\infty}(1+1/n)^{n+1} = \lim_{n\to\infty}(1+\frac{1}{n})^{n} \cdot \lim_{n\to\infty}(1+\frac{1}{n})$

$= e \cdot (1+0)$

$= e$

ii) put $n+1 = m$

$n = m-1$

$\Rightarrow \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n} = \lim_{m\to\infty}(1+\frac{1}{m})^{m-1}$

$= \lim_{m\to\infty}(1+\frac{1}{m})^{m} \cdot \lim_{m\to\infty}(1+\frac{1}{m})^{-1}$

$= \lim_{m\to\infty}\frac{(1+\frac{1}{m})^{m}}{(1+1/m)} = \frac{e}{1+0} = e$

using the identity $(1+\frac{2}{n}) = (1+\frac{1}{n+1})(1+\frac{1}{n})$, then P.T $\lim_{n\to\infty}(1+\frac{2}{n})^{n} = e^{2}$

Sol $\lim_{n\to\infty}(1+\frac{2}{n})^{n} = \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n} \cdot \left(1+\frac{1}{n}\right)^{n}$

$= e \cdot e$

$= e^{2}$

Suppose $\{s_n\}_{n=1}^{\infty}$ is the sequence of positive numbers and $0 < x < 1$. If $s_{n+1} < x s_n$ $(n \in I)$ prove that $\lim_{n\to\infty} s_n = 0$.

proof.

Given $0 < x < 1$

$\Rightarrow \lim_{n\to\infty} x^{n} = 0$ and $s_n > 0$ $(n \in I)$

also $s_{n+1} < x s_n$

$$< x^2 S_{n-1}$$

$$< x^3 S_{n-2}$$

$$< x^4 S_{n-3}$$

$$\vdots$$

$$< x^n S_1$$

NOW. $0 < S_n < x^n S_1$

$$0 < \lim_{n\to\infty} S_n < S_1 \cdot \lim_{n\to\infty} x^n$$

$$\Rightarrow 0 < \lim_{n\to\infty} S_n < S_1 (0)$$

$$\Rightarrow \lim_{n\to\infty} S_n = 0 \,//$$

1) $\lim_{n\to\infty} \frac{2n^2+5n}{4n^3+n^2} = 1/2$

Sol.
$$\lim_{n\to\infty} \frac{2n^2+5n}{4n^3+n^2} = \lim_{n\to\infty} \frac{n^3 (2+5/n)}{n^3 (4n+1/n)}$$

$$= \frac{2+0}{0+1}$$

$$= 2/1 \,//$$

2) $\lim_{n\to\infty} \frac{n^2}{(n-7)^2-6} = 1$

Sol.
$$\lim_{n\to\infty} \frac{n^2}{(n-7)^2-6} = \frac{n^2}{n^2-49+14n-6}$$

$$= \frac{n^2}{n^2+14n-55}$$

$$= \frac{n^2}{n^2\left[1+ 14/n - \frac{55}{n^2}\right]}$$

$$= \frac{1}{1+0+0}$$

$$= 1 \,//$$

Operations on divergence sequence.

Theorem:

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequence of real numbers that diverge to infinity, Then so do their sum and product.

(ie) $\{s_n + t_n\}_{n=1}^{\infty}$ and $\{s_n t_n\}_{n=1}^{\infty}$ diverges to infinity.

proof.

For every $M>0$ there exists $N_1 \in I$

Such That, $s_n \geq M$, $(n \geq N_1)$ —— (1)

Also $\{t_n\}$ diverges to ∞, there exists $N_2 \in I$,

such that

$t_n \geq 1$, $(n \geq N_2)$ —— (2)

Take $N = \max(N_1, N_2)$

Take $s_n + t_n \geq M + 1 > M$, $(n \geq N)$

and $s_n \cdot t_n \geq M \cdot 1 > M$, $(n \geq N)$

$\therefore (s_n + t_n)_{n=1}^{\infty}$ and $(s_n t_n)_{n=1}^{\infty}$ diverges to ∞.

Theorem:

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequence of real numbers if $\{s_n\}_{n=1}^{\infty}$ diverges to ∞ and if $\{t_n\}_{n=1}^{\infty}$ is bounded, Then $\{s_n + t_n\}_{n=1}^{\infty}$ diverges to ∞.

proof

Given $\{t_n\}$ is bounded, $\exists$ $Q > 0$ such that

$|t_n| \leq Q$ $(n \in I)$

Also $\{s_n\}$ diverges to ∞.

For $M > 0$, $\exists$ $N \in I$ such that

$s_n > M + Q$, $n \geq N$.

For $n \geq N$, $s_n + t_n \geq s_n - |t_n|$

$s_n + t_n > M + Q - Q$

$$s_n + t_n > M$$

$\Rightarrow (s_n + t_n)$ diverges to ∞

Corollary :

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are of real numbers, If $\{s_n\}_{n=1}^{\infty}$ diverges to ∞ and $\{t_n\}_{n=1}^{\infty}$ converges, than $\{s_n + t_n\}_{n=1}^{\infty}$ diverges to ∞.

proof. $\{t_n\}$ converges

$\Rightarrow \{t_n\}$ is bounded

$\therefore$ By the above theorem

$\{s_n + t_n\}_{n=1}^{\infty}$ diverges to ∞.

Limit Superior and Limit inferior :

NOTE :

Consider the sequence $\{s_n\}_{n=1}^{\infty}$ of real numbers which is bounded above.

Its range $= \{s_1, s_2, \ldots\}$

$M_1 = \text{l.u.b} \{s_1, s_2, \ldots\}$

$M_2 = \text{lub} \{s_2, s_3, \ldots\}$

$\vdots$

$M_n = \text{lub} \{s_n, s_{n+1}, \ldots\}$

$\vdots$

$M_1 \geq M_2 \geq \ldots \geq M_n \geq M_{n+1}$

$\therefore \{M_n\}$ is no-increasing sequence.

i) If $\{M_n\}$ is bounded below then $\lim\limits_{n\to\infty} M_n$ exists

$\therefore$ we define

$$\lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} M_n$$

iii) If $\{M_n\}$ is bounded below then

$\lim_{n\to\infty} M_n = -\infty$

we define $\lim_{n\to\infty} \sup s_n = -\infty$

consider the Sequence $\{s_n\}_n^{\infty}$ of real numbers

Which is bounded below.

If range $= \{s_1, s_2, \cdots\}$

$m_1 = g.l.b \{s_1, s_2, \cdots\}$

$m_2 = g.l.b \{s_2, s_3, \cdots\}$

$m_3 = g.l.b \{s_3, s_4, \cdots\}$

$\vdots$

$m_n = g.l.b \{s_n, s_{n+1}, \cdots\}$

$m_1 \leq m_2 \leq \cdots \leq m_n \leq m_{n+1} \leq$

$\therefore \{m_n\}$ is non-decreasing Sequence.

i) If $\{m_n\}$ bounded above

$\Rightarrow \lim_{n\to\infty} m_n$ exists

$\therefore$ we define

$\lim_{n\to\infty} \inf s_n = \lim_{n\to\infty} m_n$

ii) If $\{m_n\}$ is not bounded above,

$\Rightarrow \lim_{n\to\infty} m_n = 0$

$\therefore$ we define $\lim_{n\to\infty} \inf s_n = +\infty$

Defn: (Limit Superior)

Let $\{s_n\}_{n=1}^{\infty}$ be a Sequence of real numbers

i.e., bounded above, and let

$M_n = l.u.b \{s_n, s_{n+1}, \cdots\}$

a) If $\{M_n\}_{n=1}^{\infty}$ converges, we define

$$\lim_{n\to\infty} \sup = \lim_{n\to\infty} M_n$$

b) If $\{M_n\}_{n=1}^{\infty}$ diverges to minus infinity

i.e. $\lim_{n\to\infty} \sup S_n = -\infty$

Defn: [limit inferior]

let S_n be a sequence of real numbers is bounded below and let $m_n = g.l.b\{S_n, S_{n+1}, \cdots\}$

a) If $\{m_n\}_{n=1}^{\infty}$ converges, we define

$$\lim_{n\to\infty} \inf = \lim_{n\to\infty} m_n$$

b) If $\{m_n\}_{n=1}^{\infty}$ diverges to infinity

i.e. $\lim_{n\to\infty} \inf S_n = +\infty$

Defn:

If $\{S_n\}_{n=1}^{\infty}$ is a sequence of real numbers

is not bounded above, we write

$$\lim_{n\to\infty} \sup S_n = \infty$$

Theorem:

If $\{S_n\}_{n=1}^{\infty}$ is a convergent sequence of real numbers, Then $\lim_{n\to\infty} \sup S_n = \lim_{n\to\infty} S_n$

proof

Given $\{S_n\}_{n=1}^{\infty}$ is convergent

$\Rightarrow \lim_{n\to\infty} S_n = L$ (say)

Given $\epsilon > 0$, there exists $N \in I$ such that

$|S_n - L| < \epsilon \quad (n \geq N)$

$$\Rightarrow -\varepsilon < s_n - l < \varepsilon \quad (n \geq N)$$

$$\Rightarrow l-\varepsilon < s_n < l+\varepsilon \quad (n \geq N)$$

Thus, if $n \geq N$, then

$l+\varepsilon$ is a upper bound for $\{s_n, s_{n+1}, \ldots\}$ and

$l-\varepsilon$ is not an upper bound.

Hence $l-\varepsilon < M_n = \text{lub}\{s_n, s_{n+1}, \ldots\}$

$$\leq l+\varepsilon, \quad n \geq N$$

$$l-\varepsilon < M_n \leq l+\varepsilon \quad (n \geq N)$$

$$l-\varepsilon < \lim_{n\to\infty} M_n \leq l+\varepsilon, \quad (n \geq N)$$

$$l-\varepsilon < \lim_{n\to\infty} \sup s_n \leq l+\varepsilon, \quad (n \geq N)$$

$$-\varepsilon < \lim_{n\to\infty} \sup s_n - l < \varepsilon$$

$$\left|\lim_{n\to\infty} \sup s_n - l\right| < \varepsilon$$

Since ε is arbitrary.

$$\lim_{n\to\infty} \sup s_n = l = \lim_{n\to\infty} s_n.$$

Dfn :-

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real number that is not bounded below, we write $\lim_{n\to\infty} \inf s_n = -\infty$

Theorem : If $\{s_n\}_{n=1}^{\infty}$ is convergent

$$\Rightarrow \lim_{n\to\infty} s_n = l \text{ [Say]}$$

$\Rightarrow$ Given $\varepsilon > 0$, there exists $N \in I$ such that

$$|s_n - l| < \varepsilon \quad (n \geq N)$$

$\Rightarrow l-\epsilon < s_n < l+\epsilon \quad (n \geq N)$

Thus, if $n \geq N$, then

$l+\epsilon$ is not an lower bound for $\{s_n, s_{n+1}, \dots\}$

and $l-\epsilon$ is an lower bound.

$$\text{Hence } l-\epsilon < m_n = g.l.b\{s_n, s_{n+1}, \dots\}$$
$$\leq l+\epsilon \quad (n \geq N)$$

$l-\epsilon < m_n \leq l+\epsilon \quad (n \geq N)$

$l-\epsilon < \lim_{n\to\infty} m_n \leq l+\epsilon, \quad (n \geq N)$

$l-\epsilon < \lim_{n\to\infty} \inf s_n \leq l+\epsilon, \quad (n \geq N)$

$-\epsilon < \lim_{n\to\infty} \inf s_n - l < \epsilon$

$\left|\lim_{n\to\infty} \inf s_n - l\right| < \epsilon$

Since ϵ is arbitrary,

$$\lim_{n\to\infty} \inf s_n = l = \lim_{n\to\infty} s_n$$

NOTE :

i) If $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence of real Numbers then,

$$\lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} \inf s_n = \lim_{n\to\infty} s_n$$

ii) we make the notation for the symbols $-\infty$ and ∞ That $-\infty < x \ (x \in R)$, $x < \infty \ (x \in R)$, $-\infty < \infty$

Theorem :

If $\{s_n\}_{n=1}^{\infty}$ is a Sequence of real numbers then $\lim_{n\to\infty} \inf s_n \leq \lim_{n\to\infty} \sup s_n$.

proof :

If $\{s_n\}_{n=1}^{\infty}$ is bounded then

$$\text{g.l.b } m_n = \text{g.l.b}\{s_n, s_{n+1} \cdots\} \leq \text{l.u.b}\{s_n, s_{n+1} \cdots\} = \overset{\text{l.u.b}}{M_n}$$

$$\Rightarrow m_n \leq M_n$$

$$\lim_{n\to\infty} m_n \leq \lim_{n\to\infty} M_n$$

$$\lim_{n\to\infty} \inf s_n \leq \lim_{n\to\infty} \sup s_n .$$

If $\{s_n\}_{n=1}^{\infty}$ is not bounded, then either

$$\lim_{n\to\infty} \inf s_n = -\infty \text{ (or)}$$

$$\lim_{n\to\infty} \sup s_n = +\infty , \text{ and}$$

$$-\infty \leq \infty$$

$$\Rightarrow \lim_{n\to\infty} \inf s_n \leq \lim_{n\to\infty} \sup s_n .$$

Theorem :

If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers and if $\lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} \inf s_n = L$ where $L \in R$, then $\{s_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} s_n = L$.

Proof

$$\text{Given } L = \lim_{n\to\infty} \sup s_n$$

$$= \lim_{n\to\infty} M_n$$

$$\text{where } M_n = \text{l.u.b}\{s_n, s_{n+1} \cdots\}$$

given $\varepsilon > 0$, there exists $N_1 \in I$ such that

$$|M_n - L| < \varepsilon \quad (n \geq N_1)$$

$$M_n < L + \varepsilon \quad (n \geq N_1)$$

$\epsilon/2$ ~~ ~~ l:

$$S_n < L+\epsilon \quad (n \geq N_1) \longrightarrow (1)$$

Similarly.

$L = \lim_{n\to\infty} \inf S_n$, for some ϵ,

There exists $N_2 \in I$ such that

$$|m_n - L| < \epsilon \quad (n \geq N_2)$$

$$-\epsilon < m_n - L \quad (n \geq N_2)$$

$$L-\epsilon < m_n \quad (n \geq N_2)$$

$$L-\epsilon < S_n \quad (n \geq N_2) \longrightarrow (2)$$

Take $N = \max (N_1, N_2) \longrightarrow (3)$

From (1), (2), (3), $L-\epsilon < S_n < L+\epsilon \quad (n \geq N)$

$$\Rightarrow |S_n - L| < \epsilon \quad (n \geq N)$$

$$\Rightarrow \lim_{n\to\infty} S_n = L_{//}$$

Theorem :

If $\{S_n\}_{n=1}^{\infty}$ is a Sequence of real numbers and if $\lim_{n\to\infty} \sup S_n = \infty = \lim_{n\to\infty} \inf S_n$ then S_n diverges to infinity.

proof.

Since $\lim_{n\to\infty} \inf S_n = \infty$, for $M > 0$, there exists $N \in I$ such that $m_n > M \quad (n \geq N)$

where $m_n = g.l.b \{S_n, S_{n+1}, \dots\}$

$$S_n > M \quad (n \geq N)$$

$$\Rightarrow \lim_{n\to\infty} S_n = \infty_{//}$$

theorem

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ where the Sequence of real numbers

then

i) $\underset{n\to\infty}{lt} \sup (s_n+t_n) \leq \underset{n\to\infty}{lt} \sup s_n + \underset{n\to\infty}{lt} \sup t_n$

ii) $\underset{n\to\infty}{lt} \inf (s_n+t_n) \geq \underset{n\to\infty}{lt} \inf s_n + \underset{n\to\infty}{lt} \inf t_n$

proof

i) Let $M_n = l.u.b \{s_n, s_{n+1}, \ldots, s_k\}$

$P_n = l.u.b \{t_n, t_{n+1}, \ldots, t_k\}$

$s_k \leq M_n \ (k \geq n), \ t_k \leq P_n \ (k \geq n)$

$\Rightarrow s_k + t_k \leq M_n + P_n \ (k \geq n)$

$\Rightarrow l.u.b \{s_n+t_n, s_{n+1}+t_{n+1}, \ldots\}$

$\leq M_n + P_n$

$\Rightarrow \underset{n\to\infty}{lt} (s_n+t_n, s_{n+1}+t_{n+1}, \ldots\} \leq M_n + P_n$

$\underset{n\to\infty}{lt} \sup (s_n+t_n) \leq \underset{n\to\infty}{lt} \sup s_n + \underset{n\to\infty}{lt} \sup t_n$

ii) Let $m_n = g.l.b \{s_n, s_{n+1}, \ldots\}$

$p_n = g.l.b \{t_n, t_{n+1}, \ldots\}$

$s_k \geq m_n, \ (k \geq n)$

$t_k \geq p_n, \ (k \geq n)$

$s_k + t_k \geq m_n + p_n$

$g.l.b \{s_n+t_n, s_{n+1}+t_{n-1} \ldots\} \geq m_n + p_n$

$\underset{n\to\infty}{lt} g.l.b \{s_n+t_n, s_{n+1}+t_{n+1}, \ldots\} \geq \underset{n\to\infty}{lt} (m_n+p_n)$

$\underset{n\to\infty}{lt} \inf (s_n+t_n) \geq \underset{n\to\infty}{lt} \inf s_n + \underset{n\to\infty}{lt} \inf t_n$ //

Theorem :

Let $\{s_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers

i) If $\lim_{n\to\infty} \sup s_n = M$, then for any $\epsilon > 0$

a) $s_n < M+\epsilon$ for all but a finite number of values of n.

b) $s_n > M-\epsilon$ for infinitely many values of n.

ii) If $\lim_{n\to\infty} \inf s_n = m$, then for any $\epsilon > 0$

c) $s_n > m-\epsilon$ for all but a finite number of values of n.

d) $s_n < m+\epsilon$ for infinitely many values of n.

proof.

i) a) $\lim_{n\to\infty} \sup s_n = M$

Suppose (a) is false then for some $\epsilon > 0$, we will have $s_n > M+\epsilon$ for infinitely many values of n.

But the for any $n \in I$, $\{s_n, s_{n+1}, \ldots\}$ will contain a number $\geq M+\epsilon$

$$l.u.b\{s_n, s_{n+1}, \ldots\} \geq M+\epsilon$$

$$\lim_{n\to\infty} l.u.b\{s_n, s_{n+1}, \ldots\} \geq M+\epsilon$$

$$\Rightarrow \lim_{n\to\infty} \sup s_n \geq M+\epsilon$$

$$M \geq M+\epsilon$$

$$\Rightarrow \Leftarrow$$

$\therefore$ (a) is true

b) Suppose b) is false

Then for some $\varepsilon > 0$, $S_n < M + \varepsilon$ for only finite no. of values of 'n'.

$\Rightarrow$ If there exists $N \in I$ such that

$$S_n \leq M - \varepsilon \quad (n \geq N)$$

$$\Rightarrow \text{l.u.b} \{S_n, S_{n+1}, \ldots\} \leq M - \varepsilon$$

$$\lim_{n \to \infty} \text{l.u.b} \{S_n, S_{n+1}, \ldots\} \leq M - \varepsilon$$

$$\lim_{n \to \infty} \sup S_n \leq M - \varepsilon$$

$$M \leq M - \varepsilon$$

$$\Rightarrow \Leftarrow$$

$\therefore$ (b) is true

ii)

a) Suppose (c) is false, then for some $\varepsilon > 0$, we will have $S_n \leq m - \varepsilon$ for infinitely many values of n.

But then for any $n \in I$ $\{S_n, S_{n+1}, \ldots\}$ will contain a number $\leq m - \varepsilon$

$$\text{g.l.b} \{S_n, S_{n+1}, \ldots\} \leq m - \varepsilon$$

$$\lim_{n \to \infty} \inf S_n \leq m - \varepsilon$$

$$m \leq m - \varepsilon$$

$$\Rightarrow \Leftarrow$$

(c) is true.

d) Suppose (d) is false, then for some $\varepsilon > 0$, we will have $S_n \geq m + \varepsilon$ for finitely many values of n.

But then for any $n \in I$, $\{S_n, S_{n+1}, \ldots\}$ will contain a number $\geq m + \varepsilon$

i.e.) $S_n \geq m+\epsilon$

g.l.b $\{S_n, S_{n+1}, \cdots\} \geq m+\epsilon$

$\lim_{n\to\infty} \inf S_n \geq m+\epsilon$

$m \geq m+\epsilon$

$\Rightarrow \Leftarrow$

∴ (d) is true

Theorem:

Suppose $\{S_n\}_{n=1}^{\infty}$ be a bounded sequence.

$\lim_{n\to\infty} \sup S_n$ exists

Let $\lim_{n\to\infty} \sup S_n = M$

To construct a subsequence $\{S_{n_k}\}_{k=1}^{\infty}$ which converges to M.

For $\epsilon = 1$, $S_n > M-1$, for infinitely many value of n, [∵ by previous theorem (b)]

Let n_1 be one such value.

i.e.) $S_{n_1} > M-1$

$\|^{ly}$ There are infinitely many value of n such that $S_n > M - 1/2$

we can find, $n_2 \in I$ such that $n_2 > n_1$,

$S_{n_2} > M - 1/2$, continuing then for each integer $k > 1$, we can find $n_k \in I$ such that $n_k > n_{k-1}$ and

$$S_{n_k} > M - 1/k \longrightarrow (1)$$

$$M = \lim_{n\to\infty} \sup S_n$$

$\Rightarrow$ given $\epsilon > 0$, there exists $N \in I$ such that

$$S_n < M + \epsilon \quad (n \geq N) \longrightarrow ②$$

[by (a)]

Choose $K \in I$, so that

$$1/K < \epsilon, \; n_K > N$$

then if $k \geq K$

$$1/k < 1/K$$

$$n_k > N$$

we have $\frac{1}{k} < \epsilon$,

From ① & ②

$$M - \epsilon < S_{n_k} < M + \epsilon \quad (k \geq K)$$

$$| S_{n_k} - M | < \epsilon$$

$$\lim_{n \to \infty} S_{n_k} = M$$

Cauchy Sequence.

Dfn: Let $\{S_n\}_{n=1}^{\infty}$ be a Sequence of real numbers, then $\{S_n\}_{n=1}^{\infty}$ is called a cauchy Sequence if for any $\epsilon > 0$ There exists an $N \in I$ such that.

$$| S_m - S_n | < \epsilon, \; m, n \geq I$$

NOTE :

1. A Sequence $\{S_n\}_{n=1}^{\infty}$ is Cauchy if S_m and S_n are close together where m and n are large

2. A Sequence convergence without knowing its limit is called the Cauchy criterion.

theorem :

If the Sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ converges then $\{s_n\}_{n=1}^{\infty}$ is a cauchy sequence.

(or)

S.T the convergent sequence is a cauchy sequence

proof.

Given $\{s_n\}_{n=1}^{\infty}$ is converges.

$\Rightarrow \lim_{n\to\infty} s_n = l$ (say)

Given $\epsilon > 0$ there exists $N \in I$ such that

$|s_n - l| < \epsilon/2 \quad (n \geq N)$

If $m > N$ then

$|s_m - l| < \epsilon/2$

consider.

$$
\begin{aligned}
|s_n - s_m| &= |s_n - l + l - s_m| \\
&\leq |s_n - l| + |l - s_m| \quad (\because n, m \geq N) \\
&< \epsilon/2 + \epsilon/2 \\
&< \epsilon
\end{aligned}
$$

$\therefore \{s_n\}_{n=1}^{\infty}$ is a cauchy Sequence.

NOTE :

The converse of the above theorem need not be true.

Lemma :

If $\{s_n\}_{n=1}^{\infty}$ is a cauchy sequence of real numbers then $\{s_n\}_{n=1}^{\infty}$ is bounded.

proof

Given $\{s_n\}_{n=1}^{\infty}$ is cauchy Sequence

$\Rightarrow$ given $\epsilon = 1$, there exists $N \in I$ such that

$$|s_n - s_m| < 1, (n, m \geq N) \quad \text{—— (1)}$$

$$|s_n - s_N| < 1, (n \geq N)$$

consider $s_n = |s_n - s_N + s_N|$

$$\leq |s_n - s_N| + |s_N|$$

$$\leq 1 + |s_N| \qquad (n \geq N)$$

$$M = \max\{|s_1|, |s_2|, \ldots [s_{N-1}|\}$$

$$|s_n| < 1 + |s_N| + M, \ (n \in I)$$

$\therefore \{s_n\}$ is bounded.

Theorem :

If $\{s_n\}_{n=1}^{\infty}$ is a cauchy Sequence of real numbers

Then $\{s_n\}_{n=1}^{\infty}$ is convergent.

proof,

we know that $\{s_n\}_{n=1}^{\infty}$ converges to

$$\lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} \inf s_n$$

But always $\lim_{n\to\infty} \inf s_n \leq \lim_{n\to\infty} \sup s_n$ —— (1)

To prove :

$$\lim_{n\to\infty} \sup s_n \leq \lim_{n\to\infty} \inf s_n$$

Given $\{s_n\}_{n=1}^{\infty}$ is a cauchy Sequence of real numbers

Given $\epsilon > 0$, there exists $N \in I$ such that

$$|s_n - s_m| < \epsilon \quad (n, m \geq N)$$

$$\Rightarrow |s_n - s_N| < \epsilon/2 \quad (n \geq N)$$

$$-\varepsilon/2 < S_n - S_N < \varepsilon/2 \quad (n \geq N)$$

$$S_N - \varepsilon/2 < S_n < S_N + \varepsilon/2 \quad (n \geq N)$$

$S_N - \varepsilon/2$, $S_N + \varepsilon/2$ are lower bound and upper bound respectively for $\{S_N, S_{N+1}, \cdots\}$

For $n \geq N$, $S_N - \varepsilon/2$ and $S_N + \varepsilon/2$ are lower bounded and upper bounded for $\{S_n, S_{n+1}, \cdots\}$

Let $m_n = g.l.b\ \{S_n, S_{n+1}, \cdots\}$ $n \geq N$

$M_n = l.u.b\ \{S_n, S_{n+1}, \cdots\}$ $n \geq N$

Then

$$S_N - \varepsilon/2 \leq m_n \leq M_n \leq S_N + \varepsilon/2$$

$M_n - m_n \leq S_N + \varepsilon/2 - (S_N - \varepsilon/2)$ $[a < b < c < d$

$M_n - m_n \leq \varepsilon$ $\qquad a-d > b-c$

$b-c < a-d$

$$\lim_{n\to\infty} (M_n - m_n) \leq \varepsilon$$

$$\lim_{n\to\infty} \sup S_N - \lim_{n\to\infty} \inf S_n \leq \varepsilon$$

$$\lim_{n\to\infty} \sup S_n \leq \lim_{n\to\infty} \inf S_n + \varepsilon \quad \text{--- (2)}$$

Since ε is arbitrary

From (1) & (2)

$$\lim_{n\to\infty} \sup S_n = \lim_{n\to\infty} \inf S_n.$$

Nested Interval theorem:

Theorem:

For each $n \in I$, let $I_n = [a_n, b_n]$ be a non-empty closed bounded interval of real numbers such that

a) $I_1 \supset I_2 \supset I_3 \supset \dots \supset I_n \supset I_{n+1} \supset$ and

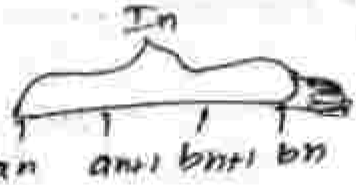
b) $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty}$ length of $I_n = 0$ then

$\bigcap_{n=1}^{\infty} I_n$ contains precisely one point.

proof

By (a) we have $I_n \supset I_{n+1}$

$\therefore a_n \le a_{n+1} \le b_{n+1} \le b_n$



$\Rightarrow \{a_n\}$ is a non-decreasing sequence

$\{b_n\}$ is non-increasing sequence

Also by (a) all terms of the sequence lies in I_1.

$\therefore$ They are bounded.

Now $\{a_n\}$ is non-decreasing sequence and bounded above.

$\Rightarrow$ It is converges.

let $\lim_{n\to\infty} a_n = x$

Also $\{b_n\}$ is non-increasing sequence and bounded below by (a)

$\Rightarrow$ It is converges

let $\lim_{n\to\infty} b_n = y$, for any $n \in I$, $a_n \le x$,

$y \le b_n \longrightarrow (1)$

Consider $y - x = \lim_{n\to\infty} b_n - \lim_{n\to\infty} b_n$

$$= \lim_{n\to\infty} (b_n - a_n)$$

$$y - x = 0 \quad [\text{by (b)}]$$

$$x = y$$

① $\Rightarrow a_n < x < b_n \,,\ n \in I$

$\Rightarrow x \in I_n \,,\ n \in I$

$\Rightarrow x \in \bigcap_{n=1}^{\infty} I_n$

Uniqueness :-

clearly no $z \neq x$ can lie in $\bigcap_{n=1}^{\infty} I_n$ Since by (b)

$|z - x| >$ length of I_n for sufficiently large n.

$$\therefore \bigcap_{n=1}^{\infty} I_n = \{x\}_{//}$$

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UNIT - 3

SERIES OF REAL NUMBERS

Introduction:

The infinite series $a_1+a_2+a_3+\ldots\ldots+a_n+\ldots\ldots$ is defined as

$\lim_{n\to\infty}(a_1+a_2+a_3+\ldots\ldots+a_n)$ provided limit exists

Definition

The infinite series $\sum_{n=1}^{\infty} a_n$ is an ordered pair $\langle \{S_n\}_{n=1}^{\infty}, \{S_n\}_{n=1}^{\infty} \rangle$, where $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers and $S_n = a_1+a_2+\ldots+a_n \ (n\in I)$

The number a_n is called the n^{th} term of the series.
The number S_n is called the n^{th} partial sum of the series.

NOTE.

1. We denote the series $\sum_{n=1}^{\infty} a_n$, (ie) $a_1+a_2+\cdots+a_n+\cdots$
$a_1+a_2+\cdots$

eg:-

$\sum (-1)^n = -1+1-1+1\cdots$

$S_n = \begin{cases} 0; & n \text{ is even} \\ -1, & n \text{ is odd} \end{cases}$

2. $\{S_n\}_{n=1}^{\infty}$. Where $S_n = a_1+a_2+\cdots+a_n$ is called the partial sum of the series $\sum_{n=1}^{\infty} a_n$

Definition

Let $\sum_{n=1}^{\infty} a_n$ be a series of real number with partial sum $S_n = a_1+a_2+\cdots+a_n$ $(n \in I)$, If the sequence $\{S_n\}_{n=1}^{\infty}$ converges to $A \in R$, We say the series $\sum_{n=1}^{\infty} a_n$ converges to A.

If $\{S_n\}_{n=1}^{\infty}$ diverges, we say $\sum_{n=1}^{\infty} a_n$ diverges.

NOTE

If $\sum_{n=1}^{\infty} a_n$ converges to A. We write.

$$\sum_{n=1}^{\infty} a_n = A$$

i.e., We use $\sum^{\infty} a_n$ to denote the series as well as its sum. Symbolically $\sum_{n=1}^{\infty} a_n = A$,

$\sum_{n=1}^{\infty} b_n = B$, $\sum (a_n+b_n) = A+B$, $C \in R$, $\sum a_n = A \Rightarrow$

$\sum Ca_n = CA$.

Theorem :-

i) If $\sum_{n=1}^{\infty} a_n$ converges to A and $\sum_{n=1}^{\infty} b_n$ converges to B, Then $\sum_{n=1}^{\infty} (a_n+b_n)$ is converges to $A+B$.

ii) Also If $c \in R$, Then $\sum_{n=1}^{\infty} c\, a_n$ converges to cA.

proof.

$$\text{Given } \sum_{n=1}^{\infty} a_n = A \; , \; \sum_{n=1}^{\infty} b_n = B$$

$$\text{Let } S_n = a_1 + a_2 + \cdots + a_n$$

$$t_n = b_1 + b_2 + \cdots + b_n$$

Then $\lim_{n\to\infty} S_n = A$, $\lim_{n\to\infty} t_n = B$

consider,

$$\sum_{k=1}^{n} (a_k + b_k) = a_1+b_1+a_2+b_2+\cdots+a_n+b_n$$

$$= (a_1+a_2+\cdots+a_n) + (b_1+b_2+\cdots b_n)$$

$$= S_n + t_n$$

$$\lim_{n\to\infty} S_n+t_n = A+B$$

$$\sum_{k=1}^{n} a_k + b_k = A+B$$

ii) Also n^{th} partial sum of

$$\sum_{k=1}^{n} c a_k = c a_1 + c a_2 + \cdots + c a_n$$

$$= c(a_1 + a_2 + \cdots + a_n)$$

$$= c\, S_n$$

$$\lim_{n\to\infty} c\, S_n = cA$$

$$\sum_{k=1}^{n} c a_k = cA .$$

Corollary :

$$\sum_{n=1}^{\infty} (a_n - b_n) = A - B$$

proof

$$\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} [a_n + (-b_n)]$$

$$= \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} (-b_n)$$

$$\therefore \sum_{n=1}^{\infty} (a_n - b_n) = A - B$$

Theorem :

If $\sum_{n=1}^{\infty} a_n$ is convergent series then $\lim_{n\to\infty} a_n = 0$

proof.

Suppose $\sum_{n=1}^{\infty} a_n = A$

Let $S_n = a_1 + a_2 + \cdots + a_n$

$\lim_{n\to\infty} S_n = A$

$a_n = S_n - S_{n-1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= A - A$$

$$\lim_{n\to\infty} a_n = 0$$

NOTE :

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is diverges.

problems :

Does the series is converges (or) diverges.

1) $\sum_{n=1}^{\infty} \frac{1-n}{1+2n}$.

let $a_n = \frac{1-n}{1+2n} = \frac{\cancel{n}\left(\frac{1}{n}-1\right)}{\cancel{n}\left(1/n+2\right)} =$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1/n - 1}{1/n+2} = -1/2 \neq 0$$

It is diverges.

2) $\sum \frac{n+3}{n+4}$

Sol: let $a_n = \frac{n+3}{n+4}$

$$= \frac{\cancel{n}(1+3/n)}{\cancel{n}(1+4/n)}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(1+3/n)}{(1+4/n)}$$

$$= 1/1$$

$$= 1 \neq 0$$

$\therefore$ It is diverges

3) $\sum \frac{n^2-2}{n^2-3}$

Sol: let $a_n = \frac{n^2-2}{n^2-3}$

$$= \frac{\cancel{n^2}(1-2/n^2)}{n^2(1-3/n^2)}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{1-2/n^2}{1-3/n^2}\right)$$

$$= 1 \neq 0$$

$\therefore$ It is diverges

NOTE :

The condition $\lim_{n\to\infty} a_n = 0$ is not sufficient for $\sum_{n=1}^{\infty} a_n$ to be convergent.

Eg : $\sum 1/n$ is not convergent, even if $\lim_{n\to\infty} 1/n = 0$.

Cauchy conditions for converges of a series

Statement :

$\sum_{k=1}^{\infty} a_k$ converges iff given $\varepsilon > 0$, there exists $N \in I$ such that $\left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon$, $n, m \geq N$

proof :

Assume $\sum_{k=1}^{\infty} a_k$ converges.

Let $S_n = a_1 + a_2 + \cdots + a_n$, $n \in I$

$\Leftrightarrow$ S_n converges.

$\Leftrightarrow$ $\{S_n\}$ is a cauchy sequences.

$\Leftrightarrow$ given $\varepsilon > 0$ there exists $N \in I$ such that

$|S_n - S_m| < \varepsilon$, $(n, m \geq N)$

$\Leftrightarrow$ if $n > m$

$|a_1 + a_2 + \cdots + a_m + a_{m+1} + a_n - (a_1 + a_2 + \cdots + a_m| < \varepsilon$, $(n, m \geq N)$

$|a_{m+1} + \cdots + a_n| < \varepsilon$

$$\left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon, \ [n, m \geq N]$$

Series with non-negative terms :

Theorem :

If $\sum_{n=1}^{\infty} a_n$ is the Series of non-negative numbers with $S_n = a_1 + a_2 + \cdots + a_n$ $(n \in I)$ then

a) $\sum_{n=1}^{\infty} a_n$ converges if the Sequence $\{s_n\}_{n=1}^{\infty}$ is bounded.

b) $\sum_{n=1}^{\infty} a_n$ diverges if $\{s_n\}_{n=1}^{\infty}$ is not bounded

proof :-

a) since $a_{n+1} \geq 0$,

we have
$$S_{n+1} = a_1 + a_2 + \cdots + a_n + a_{n+1}$$
$$= S_n + a_{n+1}$$
$$\geq S_n$$

Thus $\{S_n\}_{n=1}^{\infty}$ is non-decreasing Sequence which is bounded above is convergent.

so $\{S_n\}_{n=1}^{\infty}$ is convergent, and thus $\sum_{n=1}^{\infty} a_n$ converges

b) If $\{s_n\}_{n=1}^{\infty}$ is not bounded, then all Sequence diverges to infinity, then the Sequence is not bounded.

Theorem :-

a) If $0 < x < 1$, then $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$

b) If $x \geq 1$, then $\sum_{n=0}^{\infty} x^n$ diverges.

proof

a) $0 < x < 1$

let $S_n = 1 + x + x^2 + \cdots + x^n$

$$= \frac{1-x^{n+1}}{1-x} \qquad \left[\because S_n = a + ar + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}\right]$$

$$= \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$$

$$\lim_{n\to\infty} S_n = \frac{1}{1-x} - \lim_{n\to\infty} \frac{x^{n+1}}{1-x}$$

$$\lim_{n\to\infty} S_n = \frac{1}{1-x} - \frac{1}{1-x}(0)$$

$$\therefore \lim_{n\to\infty} S_n = \frac{1}{1-x}$$

$\therefore$ $0 < x < 1$, then $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$.

b) if $x \geq 1$

$\therefore \lim_{n\to\infty} x^n \neq 0$

$\therefore \sum_{n=0}^{\infty} x^n$ diverges.

Harmonic Series :

The Series $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ is knows as the Harmonic Series.

Theorem :

The Series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

proof:

we examine the SubSequence.

$S_1, S_2, S_4, S_8, \ldots S_{2^n} \ldots \{S_n\}_{n=1}^{\infty}$ where

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$s_1 = 1$

$s_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$$S_4 = S_2 + \frac{1}{3} + \frac{1}{4}$$

$$> \frac{3}{2} + \frac{1}{4} + \frac{1}{4}$$

$$= 2$$

$$S_8 = S_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$> 2 + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

$$S_8 = \frac{5}{2}$$

In general $S_{2^n} > \frac{n+2}{2}$

Thus $\{s_n\}_{n=1}^{\infty}$ contains a divergent subsequence & diverges.

[∵ All subsequence of a convergent sequence of real numbers converge to the same limit]

Theorem :

If $\sum_{n=1}^{\infty} a_n$ is a divergent series of positive numbers, then there is a sequence $\{\epsilon_n\}_{n=1}^{\infty}$ of the numbers which converges to zero but for which $\sum_{n=1}^{\infty} \epsilon_n a_n$ still diverges.

proof.

let $S_n = a_1 + a_2 + \dots + a_n \; (n \in I)$

To prove : $$\sum_{k=1}^{\infty} \frac{s_{k+1} - s_k}{s_{k+1}} \text{ diverges}$$

Given $\{s_n\}_{n=1}^{\infty}$ diverges to ∞

$\Rightarrow$ For any $m \in I$, choose $n \in I$ such that

$S_{n+1} > 2\, S_m$

Now $\{s_k\}_{k=1}^{\infty}$ is non-decreasing

$$\sum_{k=m}^{n} \frac{s_{k+1}-s_k}{s_{k+1}} \geq \sum_{k=m}^{n} \frac{s_{k+1}-s_k}{s_{n+1}}$$

$$\sum_{k=m}^{n} \frac{s_{k+1}-s_k}{s_{k+1}} = \frac{1}{s_{n+1}} \left[(s_{m+1}-s_m) + (s_{m+2}-s_{m+1}) + \cdots + (s_{n+1}-s_n) \right]$$

$$= \frac{1}{s_{n+1}} \left[s_{n+1} - s_m \right]$$

$$\geq \frac{1}{s_{n+1}} \left[s_{n+1} - \frac{1}{2} s_{n+1} \right]$$

$$\geq \frac{1}{2} \left[\frac{s_{n+1}}{s_{n+1}} \right]$$

$$\geq 1/2$$

thus for any $m \in I$, then there exists $n \in I$ such that

$$\sum_{k=m}^{n} \frac{s_{k+1}-s_k}{s_{k+1}} \geq 1/2$$

Thus the sequence of partial sum of the series $\sum_{k=1}^{\infty} \frac{s_{k+1}-s_k}{s_{k+1}}$ do not from a Cauchy Sequence.

$\therefore \sum_{k=1}^{\infty} \frac{s_{k+1}-s_k}{s_{k+1}}$ diverges to ∞.

$$\sum_{k=1}^{\infty} \frac{s_{k+1}-s_k}{s_{k+1}} = \infty$$

$$\sum_{k=1}^{\infty} \frac{a_k}{s_{k+1}} = \infty,$$

$$\sum_{k=1}^{\infty} \frac{a_k}{s_{k+1}} = \infty$$

let $\epsilon_k = 1/s_k$, then $\epsilon_k \longrightarrow 0$ as $k \longrightarrow \infty$

$$\Rightarrow \sum_{k=1}^{\infty} \varepsilon_k a_k = \infty ,$$

prove that the Series $\sum_{n=1}^{\infty} \left[\frac{1}{n(n+1)} \right]$ converges.

$$\left[\because \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \right]$$

Sol:

$$a_n = \frac{1}{n(n+1)}$$

$$= \frac{1}{n} - \frac{1}{n+1}$$

$$S_n = a_1 + a_2 + \cdots + a_n$$

$$= 1 + \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)$$

$$= 1$$

$\therefore \sum \frac{1}{n(n+1)}$ converges to 1.

For What Values of x, $(1-x)+(x-x^2)+\cdots$ converges.

Sol:

$$a_n = x^{n-1} - x^n$$

$$S_n = a_1 + a_2 + \cdots + a_n$$

$$= 1 - x + x - x^2 + \cdots + x^{n-1} - x^n$$

$$= 1 - x^n$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (1-x^n) = 1 - \lim_{n\to\infty} x^n$$

for $0 < x < 1$, $\left[\because \lim_{n\to\infty} x^n \text{ converges if } 0 < x < 1\right]$

Given Series is converges.

3) Does the Series $\sum \log(1+1/n)$ converges or diverges.

Sol:

let $a_n = \log(1+1/n)$

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n, \; n \in I$$

$$= \log(1+1) + \log(1+1/2) + \cdots + \log(1+1/n)$$

$$= \log 2 + \log 3/2 + \cdots + \log\left(\frac{n+1}{n}\right)$$

$$= \log\left(2 \times 3/2 \times \cdots \times \frac{n}{n-1} \times \frac{n+1}{n}\right)$$

$$= \log(n+1)$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \log(n+1)$$

$$\lim_{n\to\infty} s_n = \infty$$

$\therefore \sum \log(1+1/n)$ diverges.

Alternating Series :

Dfn.

An Alternating Series may be written as $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$.

Where a_n is positive.

i.e., $a_1 - a_2 + a_3 - a_4 + \cdots, \; a_n \geq 0, \; n \in I$.

Example :-

1) $1 - 1/2 + 1/3 - 1/4 + 1/5, \ldots$

Theorem : [Fundamental theorem of Alternating Series]

If $\{a_n\}_{n=1}^{\infty}$ is a Sequence of positive number

Such that

a) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$

i.e., $\{a_n\}_{n=1}^{\infty}$ is non-increasing

b) $\lim_{n\to\infty} a_n = 0$

Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is converges.

proof

consider first the partial sum with odd index

$$s_1, s_3, s_5, \ldots$$

$$s_1 = a_1$$

$$s_3 = a_1 - a_2 + a_3$$

$$= s_1 [-(a_2 + a_3)]$$

$$s_3 \leq s_1$$

for any n, we have

$$s_{2n+1} = s_{2n-1} [-(a_{2n} + a_{2n+1})]$$

$$\Rightarrow s_{2n+1} \leq s_{2n-1}$$

$\therefore \{s_{2n-1}\}$ is non-increasing

But $S_{2n-1} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-3} - a_{2n-2}) + a_{2n-1}$

$$s_{2n-1} > 0 \quad [\because a_{2n+1} > 0]$$

$\{s_{2n-1}\}$ is non-increasing and bounded below by a.

$\{S_{2n-1}\}$ converges.

Similarly, the sequence $s_2, s_4, \ldots, s_{2n}, \ldots$ also converges.

for $s_{2n+2} = s_{2n} + a_{2n+1} - a_{2n+2}$

$$s_{2n+2} \geq s_{2n}$$

$\{s_{2n}\}$ is non-decreasing

Also,

$$s_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - a_{2n}$$

$$s_{2n} \leq a_1 \ \forall \ (n \in I)$$

$\{s_{2n}\}$ is bounded above

$\Rightarrow \{S_{2n}\}$ converges.

let $M = \lim_{n\to\infty} S_{2n-1}$, $L = \lim_{n\to\infty} S_{2n}$

Since $a_{2n} = S_{2n-1} - S_{2n}$

$[S_{2n} = a_1 - a_2 + \cdots + a_{2n-1} - a_{2n}$

$= S_{2n-1} - a_{2n}$

$a_{2n} = S_{2n-1} - S_{2n}]$

we have by (b)

$0 = \lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} (S_{2n-1} - S_{2n})$

$= M - L$

$M = L$

Both $\{S_{2n}\}$ and $\{S_{2n-1}\}$ converges to L.

$\{S_n\}$ converges to L.

$\therefore \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges to L.

Conditional convergence and obsalute convergence:

NOTE:

$1 - 1/2 + 1/3 - 1/4 + \cdots$ converges

$1 + 1/2 + 1/3 + 1/4 + \cdots$ diverges

$1 + 1/2 + 1/4 + 1/8 \cdots$ converges to $\frac{1}{1-1/2} = 1/{1/2} = 2$.

Defn:

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers.

a) If $\sum_{n=1}^{\infty} |a_n|$ converges, we say that $\sum_{n=1}^{\infty} a_n$

converges absolutely

b) If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges,

we say that $\sum_{n=1}^{\infty} a_n$ converges conditionally.

Eg:

(a) $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \cdots$ converges.

$\therefore \sum \left|\frac{1}{2^n}\right|$ is converges.

(b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges, But $\sum_{n=1}^{\infty} \left|(-1)^{n+1} \frac{1}{n}\right|$ is diverges.

Theorem:
If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges.

proof,
Given $\sum_{n=1}^{\infty} a_n$ converges absolutely,

$\Rightarrow \sum_{n=1}^{\infty} |a_n|$ converges.

Let $S_n = a_1 + a_2 + \cdots + a_n$

$t_n = |a_1| + |a_2| + \cdots + |a_n|$

$\sum_{n=1}^{\infty} |a_n|$ converges.

$\Rightarrow \{t_n\}$ is a cauchy sequence.

$[\because \{t_n\}$ converges. $\Rightarrow \{t_n\}$ is cauchy sequence$]$

$\Rightarrow$ Given $\varepsilon > 0$, There exists $N \in I$ such that

$|t_n - t_m| < \varepsilon, \; n, m \geq N$

consider : $n > m$

$|S_n - S_m| = |a_1 + \cdots + a_m + a_{m+1} + \cdots + a_n - (a_1 + \cdots + a_m)|$

$= |a_{m+1} + \cdots + a_n|$

$\leq |a_{m+1}| + \cdots + |a_n|$

$\leq |t_n - t_m|$

$< \epsilon \ , \quad n, m \geq N$

$\Rightarrow \{s_n\}$ is a cauchy Sequence

$\Rightarrow \{s_n\}$ is converges.

$\sum_{n=1}^{\infty} a_n$ is converges.

Theorem :

a) If $\sum_{n=1}^{\infty} a_n$ converges absoultely , Then both $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ converges However

b) If $\sum_{n=1}^{\infty} a_n$ converges condi'tionationally , then both $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ diverges.

Where $2p_n = a_n + |a_n|$, $2q_n = a_n - |a_n|$

proof

a) $\sum_{n=1}^{\infty} a_n$ converges absoulutely.

$\Rightarrow \sum_{n=1}^{\infty} |a_n|$ converges and $\sum_{n=1}^{\infty} a_n$ converges.

$2p_n = a_n + |a_n| \longrightarrow (1)$

$2q_n = a_n - |a_n| \longrightarrow (2)$

w.k. $p_n = \frac{a_n + |a_n|}{2}$, $q_n = \frac{a_n - |a_n|}{2}$

$\sum_{n=1}^{\infty} p_n = \frac{1}{2}\left[\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} |a_n|\right]$ converges,

$\sum_{n=1}^{\infty} q_n = \frac{1}{2}\left[\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} |a_n|\right]$ converges.

$\therefore \sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ converges.

b) $\sum_{n=1}^{\infty} a_n$ converges conditionally

$\Rightarrow \sum_{n=1}^{\infty} |a_n|$ diverges.

To prove that $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ diverges

on the contrary assume that $\sum_{n=1}^{\infty} p_n$ & $\sum_{n=1}^{\infty} q_n$ converges.

(1) $\Rightarrow$ $|a_n| = 2p_n - a_n$

$\sum_{n=1}^{\infty} 2p_n$ converges, $\sum_{n=1}^{\infty} a_n$ converges.

$\Rightarrow \sum_{n=1}^{\infty} |a_n|$ converges.

Which is a contradiction.

$\therefore \sum_{n=1}^{\infty} p_n$ diverges

$|||^{ly}$ using $|a_n| = a_n - 2q_n$ Which is also condition.

$\therefore \sum_{n=1}^{\infty} q_n$ diverges.

TESTS FOR ABSOLOTE CONVERGES

Dfn :

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of real numbers.

we shall say that $\sum_{n=1}^{\infty} a_n$ is dominated by $\sum_{n=1}^{\infty} b_n$, if there exists $N \in \mathbb{I}$ such that.

$$|a_n| \leq |b_n| \quad (n \geq N)$$

[ie) $|a_n| \leq |b_n|$ except for a finite number of values of n]

we write $\sum_{n=1}^{\infty} a_n \ll \sum_{n=1}^{\infty} b_n$.

Comparison test for absoulte converges

Statement:

If $\sum_{n=1}^{\infty} a_n$ is dominated by $\sum_{n=1}^{\infty} b_n$ where $\sum_{n=1}^{\infty} b_n$ converges absolute then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Symbolically : if $\sum_{n=1}^{\infty} a_n \ll \sum_{n=1}^{\infty} b_n$, and $\sum_{n=1}^{\infty} |b_n| < \infty$, then $\sum_{n=1}^{\infty} |a_n| < \infty$.

NOTE :

1. If $\sum_{n=1}^{\infty} a_n$ converges, we write $\sum_{n=1}^{\infty} a_n < \infty$.
2. If $\sum_{n=1}^{\infty} |a_n|$ converges, we write $\sum_{n=1}^{\infty} |a_n| < \infty$
3. If $\sum_{n=1}^{\infty} |a_n|$ diverges, we write $\sum_{n=1}^{\infty} |a_n| = \infty$

proof.

Given $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$

$\Rightarrow$ then there exists $N \in I$ such that

$$|a_n| \leq |b_n|, \ (n \geq N)$$

and also $\sum_{n=1}^{\infty} b_n$ converges absolutely.

$$\Rightarrow \sum_{n=1}^{\infty} |b_n| \text{ converges.}$$

Let $\sum |b_n| = M$

Let $S_n = |a_1| + |a_2| + \cdots + |a_n|$

$$= |a_1| + \cdots + |a_N| + |a_{N+1}| + \cdots + |a_n|$$

$$\leq |a_1| + \cdots + |a_N| + |b_{N+1}| + \cdots + |b_n|$$

$$S_n \leq |a_1| + |a_2| + \cdots + |a_N| + M, \ (n \in I)$$

$\Rightarrow \{S_n\}$ is bounded

The Sequence of partial series of $\sum_{n=1}^{\infty} |a_n|$ is

then bounded above.

$$\Rightarrow \sum_{n=1}^{\infty} |a_n| \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges absolutely.}$$

Comparison test for of divergences

Statement :

If $\sum_{n=1}^{\infty} a_n$ is dominated by $\sum_{n=1}^{\infty} b_n$ and

$\sum_{n=1}^{\infty} |a_n| = \infty$, then $\sum_{n=1}^{\infty} |b_n| = \infty$.

[ii] if $\sum_{n=1}^{\infty} a_n \sim \sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} |a_n| = \infty$ then $\sum_{n=1}^{\infty} |b_n| = \infty$]

proof.

Let $S_n = |a_1| + |a_2| + \cdots + |a_n|$

$t_n = |b_1| + |b_2| + \cdots + |b_n|$

$$\sum_{n=1}^{\infty} |a_n| = \infty$$

$\Rightarrow \{S_n\}_{n=1}^{\infty}$ is unbounded $\longrightarrow$ ①

$$\sum_{n=1}^{\infty} a_n \sim \sum_{n=1}^{\infty} b_n$$

$\Rightarrow$ then there exists $N \in I$ such that

$|a_n| \leq |b_n|$, $(n \in I)$

by ① $\{t_n\}_{n=1}^{\infty}$ is unbounded.

$$\Rightarrow \sum_{n=1}^{\infty} |b_n| = \infty,$$

Theorem:

a) If $\sum_{n=1}^{\infty} b_n$ converges absolutely and if $\lim_{n\to\infty} \frac{|a_n|}{|b_n|}$ exists

Then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

b) If $\sum_{n=1}^{\infty} |a_n| = \infty$ and if $\lim_{n\to\infty} \frac{|a_n|}{|b_n|}$ exists, then $\sum_{n=1}^{\infty} |b_n| = \infty$.

proof.

$\lim_{n\to\infty} \frac{|a_n|}{|b_n|}$ exists

$\Rightarrow \left\{ \frac{|a_n|}{|b_n|} \right\}$ converges.

$\Rightarrow \left\{ \frac{|a_n|}{|b_n|} \right\}$ is bounded.

$\Rightarrow$ Then there exists $M > 0$ such that $\frac{|a_n|}{|b_n|} \leq M$, $\forall\, n \in I$

$$|a_n| \leq M\, |b_n| \;\forall\, n \in I \longrightarrow ①$$

a) $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} M b_n$

given $\sum_{n=1}^{\infty} b_n$ converges absolutely

$\Rightarrow \sum_{n=1}^{\infty} M b_n$ converges absolutely

$\therefore$ By comparison test for convergence

$\sum_{n=1}^{\infty} a_n$ converges absolutely.

b) $① \Rightarrow \frac{1}{M} |a_n| \leq |b_n|$, $(\forall\, n \in I)$

$$\sum_{n=1}^{\infty} \tfrac{1}{M} |a_n| \leq \sum_{n=1}^{\infty} b_n$$

given $\sum_{n=1}^{\infty} |b_n| = \infty$

$\Rightarrow \sum_{n=1}^{\infty} \tfrac{1}{M} |a_n| = \infty$

By comparison test for divergence $\sum_{n=1}^{\infty} |b_n| = \infty$.

Ratio test.

If $\sum_{n=1}^{\infty} a_n$ is a Series of non-zero real numbers and

let $a = \lim_{n\to\infty} \inf \left|\frac{a_{n+1}}{a_n}\right|$, $A = \lim_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right|$ (so that $a \leq A$)

Then

a) If $A < 1$, then $\sum_{n=1}^{\infty} |a_n| < \infty$

b) If $a > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(ii) If $a < 1 \leq A$, then the test fails.

proof:

If $A < 1$ choose B such that $A < B < 1$

$$A = \lim_{n\to\infty} \sup \left| \frac{a_{n+1}}{a_n} \right|$$

$$B = A + \varepsilon, \text{ for some } \varepsilon > 0$$

By the theorem,

Let $\{s_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers.

If $\lim_{n\to\infty} \inf s_n = M$, then for any $\varepsilon > 0$.

$s_n > M + \varepsilon$ for all but a finite numbers of values of n,

a) Then there exists $N \in I$ such that

$$\left| \frac{a_{n+1}}{a_n} \right| < B \quad (n \geq N)$$

Then $\left| \dfrac{a_{N+1}}{a_N} \right| < B$

$$\left| \frac{a_{N+2}}{a_N} \right| < B$$

$$\left| \frac{a_{N+2}}{a_N} \right| = \left| \frac{a_{N+2}}{a_{N+1}} \right| \left| \frac{a_{N+1}}{a_N} \right|$$

$$< B B$$

$$< B^2$$

For any $k \geq 0$

$$\left| \frac{a_{N+k}}{a_N} \right| = \left| \frac{a_{N+k}}{a_{N+k-1}} \right| \left| \frac{a_{N+k-1}}{a_{N+k-2}} \right| \cdots \left| \frac{a_{N+1}}{a_N} \right|$$

$$= B \cdot B \cdots B$$

$$< B^k$$

$|a_{N+k}| < B^k |a_N|, \; (k = 0, 1, 2 \ldots$

$\Rightarrow \sum a_{N+k} < \sum |a_N| B^k$

Since $0 < B < 1$

$\sum_{k=0}^{\infty} |a_N| B^k$ converges

$\Rightarrow \sum_{k=0}^{\infty} |a_{N+k}|$ converges

$\Rightarrow |a_N| + |a_{N+1}| + \cdots$ converges.

$\Rightarrow |a_1| + |a_2| + \cdots + |a_N| + |a_{N+1}| + \cdots$ converges

b) If $a > 1$, $\lim_{n\to\infty} \inf \left|\frac{a_{n+1}}{a_n}\right| > 1$

By the theorem.

$\lim_{n\to\infty} \inf S_n = m$, then for any $\epsilon > 0$,
$S_n > m - \epsilon$ for all but a finite number of

Values of n.

$\left|\frac{a_{n+1}}{a_n}\right| > 1 \quad (\forall\, n \geq N)$

$\left|\frac{a_{N+1}}{a_N}\right| > 1$

$|a_{N+1}| > |a_N|$

$|a_N| < |a_{N+1}| < |a_{N+2}| < \cdots$

$\lim_{n\to\infty} |a_n| \neq 0$, $\sum_{n=1}^{\infty} a_n = \infty$

i.e) $\{a_n\}_{n=1}^{\infty}$ does not converges to zero

$\therefore \sum_{n=1}^{\infty} a_n$ is diverge.

c) $a = A = 1$

Let $a_n = \frac{1}{n}$, $a_{n+1} = \frac{1}{n+1}$

$$\frac{a_{n+1}}{a_n} = \frac{1}{n+1} \cdot \frac{n}{1}$$

$$= \frac{n}{n+1}$$

$$= \frac{\cancel{n}}{\cancel{n}(1+1/n)}$$

$$a = A = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$$

$\sum_{n=1}^{\infty} 1/n = \infty$ diverges

$$a_n = 1/n^2 , \quad a_{n+1} = \frac{1}{(n+1)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} = \frac{n^2}{n^2(1+\frac{1}{n^2})}$$

$$a = A = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$$

But $\sum 1/n^2$ converges

$\therefore$ test fails.

problems.

8.7 $\sum \frac{n^n}{n!}$ diverges

proof

$$a_n = \frac{n^n}{n!} , \quad a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$$

$$= \left(\frac{n+1}{n}\right)^n$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{n(1+1/n)}{\cancel{n}}\right]^n$$

$$= e > 2$$

$\therefore \; a = A = e > 2$

$\Rightarrow \sum \frac{n^n}{n!}$ is diverges

S.T $\sum \frac{n!}{n^n}$ converges.

Soln

$$a_n = \frac{n!}{n^n}, \quad a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!}$$

$$= \left(\frac{n}{n+1}\right)^n$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{\cancel{n}}{\cancel{n}(1+1/n)}\right]^n$$

$$= 1/e < 1/2 < 1$$

$$\therefore \sum \frac{n!}{n^n} \text{ is converges}$$

Root test :

Statement

If $\lim_{n\to\infty} \sup \sqrt[n]{|a_n|} = A$, then the series of real numbers

$$\sum_{n=1}^{\infty} a_n$$

a) converges absolutely if $A < 1$.

b) diverges if $A > 1$.

c) If $A = 1$, the test fails

proof.

a) If $A < 1$, choose B such that $A < B < 1$, Then there exists $N \in I$ such that $\sqrt[n]{|a_n|} < B \quad (n \geq N)$

$$|a_n| < B^n \quad (n \geq N)$$

$$\sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} B^n$$

But $0 < B < 1$

$\Rightarrow \sum B^n$ converges absolutely

$\Rightarrow \sum |a_n| < \infty$ (By comparison test)

b) If $A > 1$, $\lim_{n\to\infty} \sup \sqrt[n]{|a_n|} > 1$

$\Rightarrow$ Then there exists $N \in I$ such that

$$\sqrt[n]{|a_n|} > 1, \quad n \geq N$$

$$|a_n| > 1$$

$$\lim_{n\to\infty} a_n \neq 0$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

c) $A = 1$, let $a_n = 1/n$

$$\sqrt[n]{|a_n|} = (1/n)^{1/n}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} (1/n)^{1/n}$$

$$= \lim_{n\to\infty} e^{\log(1/n)^{1/n}}$$

$$= \lim_{n\to\infty} e^{1/n(-\log n)}$$

$$= e^{-\left[\lim_{n\to\infty} 1/n \cdot \lim_{n\to\infty} \log n\right]}$$

$$= e^{-0}$$

$$= 1$$

If $a = n = 1$

let $a_n = 1/n^2$

$$\sqrt[n]{|a_n|} = (1/n^2)^{1/n}$$

$$= (1/n)^{2/n}$$

$$= \left[\left(\frac{1}{n}\right)^{1/n}\right]^2$$

$$= \left[\lim_{n\to\infty}\left(\frac{1}{n}\right)^{1/n}\right]^2$$

$$= 1$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

$\therefore$ Test fails.

Do the following Series Converge:

1) $\sum_{n=1}^{\infty} \frac{n^4}{n!}$

Sol let $a_n = \frac{n^4}{n!}$

$$a_{n+1} = \frac{(n+1)^4}{(n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^4}{(n+1)!} \times \frac{n!}{n^4}$$

$$= \frac{(n+1)^3}{n^4}$$

$$= \frac{\cancel{n^3}(1+1/n)^3}{n^{\cancel{4}}}$$

$$= \frac{(1+1/n)^3}{n}$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{n}(1+1/n)^3$$

$$a = A = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{n}(1+1/n)^3$$

$$= 0 < 1$$

$\therefore$ The Series Converges absolutely by ratio test

2) $\lim_{n\to\infty} \frac{1+n}{1+n^2}$

Sol

Let $a_n = 1/n$, $b_n = \frac{1+n}{1+n^2}$

$a_1 = 1$, $b_1 = 1$

$a_2 = 1/2$, $b_2 = 3/5$

$|a_n| \leq |b_n| \forall n$

$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$

But $\sum_{n=1}^{\infty} 1/n = \infty$

$\therefore$ By Comparison test the Series is diverges

$\sum_{n=1}^{\infty} \frac{1+n}{1+n^2} = \infty$

3) $\sum_{n=1}^{\infty} \frac{3}{4+2^n}$

Sol.

Let $a_n = \frac{3}{4+2^n}$, $b_n = \frac{3}{2^n}$

$a_1 = 3/6 = 1/2$; $b_1 = 3/2$

$a_2 = 3/8$; $b_2 = 3/4$

$|a_n| \leq |b_n| \forall n$

$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$

But $\sum_{n=1}^{\infty} 3/2^n = 3\sum_{n=1}^{\infty} 1/2^n$ is converges

absolutely by Comparison test

$|||^{ly}$ $\sum_{n=1}^{\infty} \frac{3}{4+2^n}$ converges absolutely.

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UNIT - IV

Series whose terms from a non-increasing sequence

Cauchy condensation test :

If $\{a_n\}_{n=1}^{\infty}$ is an non-increasing seq. of postive numbers and if $\sum_{n=1}^{\infty} 2^n . a_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

Proof :-

we have $a_1 \le a_1 \le 2^0 a_{2^0}$

$a_2 + a_3 \le a_2 + a_2 = 2a_2$

$a_4 + a_5 + a_6 + a_7 \le a_4 + a_4 + a_4 + a_4 = 4a_4 = 2^2 a_{2^2}$

and for any $n \in I$

$a_{2^n} + a_{2^n+1} + a_{2^n+2} + \dots + a_{2^{n+1}-1} \le 2^n a_{2^n}$

from this, we have,

$$\sum_{k=1}^{2^{n+1}-1} a_k \le \sum_{k=0}^{n} 2^k a_{2^k}$$

$$\le \sum_{k=0}^{\infty} a^k a_{2^k}$$

ie) for any $n \in I$, we have.

$$\sum_{k=1}^{\infty} a_k \le \sum_{k=0}^{\infty} a^k . a_{2^k} \longrightarrow ①$$

given .

$$\sum_{k=1}^{\infty} 2^k a_{2^k} \text{ converges}$$

Let $\sum_{k=1}^{\infty} 2^k a_{2^k} = L$

$① \Rightarrow \sum_{k=1}^{\infty} a_k \le L$

ie, seq. of Partial

sum of $\sum_{n=1}^{\infty} a_n$ is bounded

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

Note :

The converse is also true .

Cauchy condensation test of a diverges of a series

Theorem :

If $\{a_n\}_{n=1}^{\infty}$ is non - increasing seq of positive number and if $\sum_{n=1}^{\infty} 2^n a_{2^n}$ diverges , $\sum_{n=1}^{\infty} a_n$ diverges

Proof :-

we have ,

$$a_3 + a_4 \geq a_4 + a_4 = 2a_4 = \frac{1}{2}(2^2 a_{2^2})$$

$$a_5 + a_6 + a_7 + a_8 \geq 4a_8 = \frac{1}{2}(2^3 a_{2^3})$$

In general

$$a_{2^n+1} + a_{2^n+2} + \cdots\cdots + a_{2^{n+1}} \geq 2^n a_{2^{n+1}}$$

$$= \frac{1}{2}(2^{n+1} a_{2^{n+1}})$$

so that $\sum_{k=2}^{2^{n+1}} a_k \geq \frac{1}{2} \sum_{k=2}^{n+1} 2^k a_{2^k}$

$\therefore$ seq of partial sum of Σa_n is not bounded

Corollary :

The series $\sum_{n=1}^{\infty} 1/n^2$ converges

soln :-

Let $a_n = 1/n^2$

$$a_{2^n} = \frac{1}{(2^n)^2} = \frac{1}{2^{2n}}$$

$$\sum_{n=1} 2^n a_{2^n} = \sum_{n=1} \frac{1}{2^n} < \infty$$

By Cauchy condensation test $\sum_{n=1}^{\infty} a_n < \infty$

Corollary :-

$$\sum_{n=1}^{\infty} 1/n \text{ converges}$$

soln:-

Let $a_n = 1/n$, $a_{2^n} = 1/2^n$

$$2^n a_{2^n} = 2^n/2^n = 1$$

$$\sum_{n=1} 2^n a_{2^n} = \sum_{n=1} 1. = \infty$$

By Cauchy condensation test

$$\sum_{n=1} a_n = \infty$$

$$\therefore \sum_{n=1} 1/n = \infty$$

① $\sum_{n=4}^{\infty} \frac{1}{n \log n}$

soln:-

Let $a_n = \frac{1}{n \log n}$

$$a_{2^n} = \frac{1}{2^n \log 2^n}$$

$$2^n a_{2^n} = \cancel{2^n}/\cancel{2^n} \log 2^n$$

$$= \frac{1}{\log 2^n}$$

$$\sum 2^n a_{2^n} = \frac{1}{n \log 2}$$

$$= \log 2 \sum 1/n$$

$$= \infty$$

$\therefore$ By cauchy condensation test,

$$\sum \frac{1}{n \log n} = \infty$$

Pringsheim's theorem :-

If $\{a_n\}$ is a non-increasing seq of positiver number of $\sum_{n=1}^{\infty} a_n$ converges then $\lt_{n\to\infty} na_n = 0$.

proof:-

Let $S_n = a_1+a_2+\cdots+a_n$, $n\in I$

$$\sum_{n=1}^{\infty} a_n = n$$

$$\lt_{n\to\infty} S_n = A = \lt_{n\to\infty} S_{2n}$$

$$S_{2n} - S_n = a_{n+1} + \cdots + a_{2n}$$

$$\geq a_{2n} + a_{2n} + \cdots + a_{2n}$$

$$\geq n a_{2n}$$

$$\geq 1/2 \; 2n \, a_{2n}$$

$$0 \leq 1/2\,(2n a_{2n})$$

$$\leq S_{2n} - S_n$$

Taking $\lt_{n\to\infty}$

$$0 \leq 1/2 \lt_{n\to\infty} 2na_{2n} \leq 0$$

$$\Rightarrow \lt_{n\to\infty} 2na_{2n} \leq 0$$

$$\Rightarrow \lt_{n\to\infty} 2na_{2n} = 0$$

But $a_{n+1} \leq a_{2n}$

$$0 \leq (2n+1)\, a_{2n+1}$$

$$\leq \frac{2n+1}{2n} \cdot 2n a_{2n}$$

Taking $\lt_{n\to\infty} (2n+1) a_{2n+1}$ $\left[\because \lt_{n\to\infty}\left(1+\frac{1}{2n}\right)^{2n} a_{2n} = 0\right]$

$$\lt_{n\to\infty} (2n+1) a_{2n+1} = 0$$

Note :-

The above theorem is not true. if we drop the hypothesis that $\{a_n\}$ is non-increasing.

Limits and metric spaces.

Def : limit of a function on a real line :

we say that $F(x)$ approaches L, (where $L \in R$) as x approaches a, if given $\epsilon > 0$ $\exists$ a $\delta > 0$ such that,

$$0 < |x-a| < \delta \Rightarrow |F(x) - L| < \epsilon$$

(or)

$|F(x) - L| < \delta$ we write $\lim_{x \to a} F(x) = L$

Theorem :

If $\lim_{x \to a} F(x) = L$ and $\lim_{x \to a} g(x) = M$ Then

$\lim_{x \to a} F(x) + g(x)$ has a limit as $x \to a$

(ie) $\lim_{x \to a} F(x) + g(x) = L + M$

(or)

limit of sum of two functions is sum of their limits.

proof :

Given $\lim_{x \to a} F(x) = L$, $\lim_{x \to a} g(x) = M$

Given, $\epsilon > 0$, $\exists$ $\delta_1 > 0$ such that

$0 < |x-a| < \delta_1 \Rightarrow |F(x) - L| < \epsilon/2$ and

$\delta_2 > 0$ such that

$0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \epsilon/2$

Take $\delta = \min(\delta_1, \delta_2)$. Then $0 < |x-a| < \delta \Rightarrow |F(x) - L| < \epsilon/2$ and $|g(x) - M| < \epsilon/2$

Consider

$$|f(x)+g(x)-(L+M)| = |f(x)-L| + |g(x)-M|$$
$$\le |f(x)-L| + |g(x)-M|$$
$$\le \epsilon/2 + \epsilon/2$$
$$< \epsilon \text{ if } 0<|x-a|<\delta$$

$\therefore \underset{x\to a}{lt} [f(x)-g(x)] = L+M$

Theorem :-

If $\underset{x\to a}{lt} f(x) = L$ and $\underset{x\to a}{lt} g(x) = M$ Then

(a) $\underset{x\to a}{lt} f(x)-g(x) = L-M$ (b) $\underset{x\to a}{lt} f(x)\cdot g(x) = LM$

and if $M \neq 0$ (c) $\underset{x\to a}{lt} \dfrac{f(x)}{g(x)} = L/M$

Proof :-

(a) consider $|f(x)-g(x)-(L-M)| = |f(x)-L-(g(x)-M)|$

$$= |f(x)-L| + |-(g(x-M)|$$
$$\le |f(x)-L| + |g(x)-M|$$
$$\le \epsilon/2 + \epsilon/2$$
$$< \epsilon \quad \text{if } 0<|x-a|<\delta$$

$\Rightarrow \underset{x\to a}{lt} [f(x)-g(x)] = L-M$

b) consider

$$|f(x)g(x)-LM| = |f(x)g(x)-Lg(x)+Lg(x)-LM|$$
$$= |g(x)(f(x)-L)+L(g(x)-M)|$$
$$\le |g(x)||f(x)-L| + |L||g(x)-M| \to ①$$

Given $\underset{x\to \infty}{lt} g(x) = M$

Given $\lim_{n \to \infty} g(x) = M$

Given, $\epsilon = 1$, $\exists\ \delta_1 > 0$ such that $0 < |x-a| < \delta_1$

$\Rightarrow |a(x) - M| < 1$

$|g(x)| < |M| + 1 = A$ (say) $\rightarrow$ ②

Also $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$

Given $\epsilon > 0$ $\exists$ a $\delta_2 > 0$ such that $0 < |x-a| < \delta_2$

$\Rightarrow |f(x) - L| < \epsilon/2A \rightarrow$ ③

$\exists\ \delta_3 > 0$, such that $0 < |x-a| < \delta_3$

$\Rightarrow |g(a) - M| < \epsilon/2|L|$

Take $\delta = \min(\delta_1; \delta_2, \delta_3)$

Then for $0 < |x-a| < \delta$

from ②③④ in ①

① $\Rightarrow |f(x)\,g(x) - LM| < \epsilon/2A \cdot A + \frac{\epsilon}{2|L|} \cdot |L|$

$< \epsilon$

$\therefore \lim_{x \to a} f(x)\,g(x) = Lm.$

(c) we shall prove a Lemma $m \neq 0$

$\lim_{x \to a} g(x) = M \Rightarrow \lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}$

(i) Assume $M > 0$

consider $\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \left| \frac{(m - g(x))}{g(x)M} \right|$

$= \frac{|g(x) - M|}{|g(x)M|} \rightarrow$ ①

Given $\lim_{x \to a} g(x) = M$

Given, $\epsilon = M/2$ $\exists$ a $\delta_1 > 0$, such that $0 < |x-a| < \delta_1$

$\Rightarrow |g(x) - M| < M/2$

$\Rightarrow -M/2 < g(x) - M < M/2$

$\Rightarrow M/2 < g(x) < 3M/2 \Rightarrow M/2 < |g(x)| \rightarrow (2)$

Also for $\epsilon > 0$, $\exists\ \delta_2 > 0$ such that $0 < |x-a| < \delta_2 \Rightarrow$

$|g(x) - M| < M^2/2\ \epsilon \rightarrow (3)$

Take $\delta = \min(\delta_1, \delta_2)$

Then for $0 < |x-a| < \delta$.

we have from (1).

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| < \frac{\frac{M^2\epsilon}{2}}{M/2 \cdot M}$$

$$< \epsilon$$

$$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}$$

(ii) If $M < 0$ replacing

M by $-M$ and $g(x)$ by $-g(x)$

in the above proof.

$\therefore M \neq 0$

$\lim_{x \to a} g(x) = M$

$\Rightarrow \lim_{x \to a} \frac{1}{g(x)} = 1/M$

Now consider,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

$$= L \cdot \frac{1}{M}$$

$$= L/M$$

Def :

we say that $f(x)$ approaches L as x approaches ∞. if Given $\epsilon > 0$, $\exists$ $M \in \mathbb{R}$ such that $|f(x) - L| < \epsilon$, $(x > M)$. and we write

$$\lim_{x \to \infty} f(x) = L.$$

Defn :

we say that $f(x)$ approaches L as x approaches 'a' from the right. if Given $\epsilon > 0$, $\exists$ $\delta > 0$, such that $a < x < a + \delta$

$\Rightarrow |f(x) - L| < \epsilon$

we write $\lim_{x \to a^+} f(x) = L$ [L is called the right hand limit of $f(x)$ at a]

we say that $f(x)$ approaches L as x, approaches a from the left. if given $\epsilon > 0$ $\exists$ $\delta > 0$ such that $a - \delta < x < a \Rightarrow |f(x) - L| < \epsilon$.

we write $\lim_{x \to a^-} f(x) = L$

(L is called the left hand limit of $f(x)$ at a)

Eg :-

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 3 - x & 1 \le x \le 2 \end{cases}$$

$\lim_{x \to 1^-} f(x) = 1$ $\quad$ $\lim_{x \to 1^+} f(x) = 2.$

Defn:-

Non - decreasing function :

If f is real value function on an interval $J \subset R$ we say that f is non - decreasing on J.

If $x < y$, $x, y \in J \Rightarrow f(x) \leq f(y)$

Non - increasing function : If f is real valued function on an interval $J \subset R$, we say that f is non-increasing on S if $x > y$, $x, y \in J$.

Bounded function :

we say that a function 'f' on an interval $J \subset R$ is bounded above and bounded below. If the range of 'f' is respectively bounded above and below

If the range is bounded then the function is bounded.

Theorem :

Let f be a non-decreasing function on the bounded open interval (a, b).

(i) If 'f' is bounded above on (a, b) then $\underset{x \to b}{lt} - f(x)$ exists.

(ii) Also if 'f' is bounded below on (a, b) Then $\underset{x \to a^+}{lt} f(x)$ exists.

Proof:-

(i) f is bounded above and non-decreasing on (a, b)

$\therefore$ $M = lub\ f(x)$, $x \in (a, b) \rightarrow ①$

Given $\epsilon > 0$, $M - \epsilon$ is not an unbounded for the range of 'f'.

$\therefore \exists\ y \in (a,b)$ such that $f(y) > M - \epsilon$

Let $\delta = b - y \Rightarrow y = b - \delta$

$$f(b-\delta) = f(y) > M - \epsilon \rightarrow (2)$$

Since f is non-decreasing.

$$f(x) > M - \epsilon \rightarrow (3) \quad (b - \delta < x < b)$$

But by (1) $f(x) \leq M$, $\forall x \in (a,b)$

$$f(x) < M + \epsilon, \ \forall x \in (a,b) \rightarrow (4)$$

From (3) & (4)

$$\Rightarrow M - \epsilon < f(x) < M + \epsilon \quad (b - \delta < x < b)$$

$$|f(x) - M| < \epsilon \quad (b - \delta < x < b)$$

$$\Rightarrow \lim_{x \to b-} f(x) = M$$

(ii) f is bounded below a similar argument will show that show that $\lim_{x \to a+} f(x) = m$,

where $m = \text{glb } f(x) \ x \in (a,b)$

The Class l^2 :-

Defn:-

The Class l^2 is the Class of all sequences $S = \{S_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} S_n^2 < \delta$.

Thus the element of l^2 are sequence

Eg:

The seq $0, 0, 0 \ldots$ is clearly an element of l^2.

Thearam : [The schewarz Inequality]

If $s = \{s_n\}_{n=1}^{\infty}$ and $t = \{t_n\}_{n=1}^{\infty}$ are in ℓ^2 the $\sum_{n=1}^{\infty} s_n t_n$ is absolutely converges and,

$$\left|\sum_{n=1}^{\infty} s_n t_n\right| \le \left(\sum_{n=1}^{\infty} s_n^2\right)^{1/2} \left(\sum_{n=1}^{\infty} t_n^2\right)^{1/2} \rightarrow ①$$

Proof:-

Let us assume that atleast one s_n, say $s_n \neq 0$, atherwise the thearam is trivial for find $n \ge N$ and any $x \in R$, we have $\sum_{k=1}^{n} (x s_k + t_k)^2 \ge 0$

$$x^2 \sum_{k=1}^{n} s_k^2 + 2x \sum_{k=1}^{n} s_k t_k + \sum_{k=1}^{n} t_k^2 = 0$$

This is of the from,

$$Ax^2 + Bx + C \ge 0, \text{ we have}$$

$$A = \sum_{k=1}^{n} s_k^2 \ge 0, \quad B = \sum_{k=1}^{n} s_k t_k, \quad C = \sum_{k=1}^{n} t_k^2$$

we knaw the minimum value of

$$Ax^2 + Bx + C \ge 0 \quad (A > 0)$$

accur, when,

$$x = -B/2A$$

put $x = -B/2A$,

$$A\left(-B/2A\right)^2 + B\left(-B/2A\right) + C \ge 0$$

$B^2 \le 4AC$, this says that

$$\text{(ie)} \left(\sum_{k=1}^{n} s_k t_k\right)^2 \le \left(\sum_{k=1}^{n} s_k^2\right)\left(\sum_{k=1}^{n} t_k^2\right) \rightarrow ②$$

Replacing s_k, t_k by,

$|s_k|, |t_k|$ in ②

$$\sum_{k=1}^{n} |s_k t_k| \leq \left(\sum_{k=1}^{k} s_k^2\right)^{1/2} \left(\sum_{k=1}^{n} t_k^2\right)^{1/2}$$

$$\leq \left(\sum_{k=1}^{\infty} s_k^2\right)^{1/2} \cdot \left(\sum_{k=1}^{\infty} t_k^2\right)^{1/2}$$

Thus the seq of partial sum of,

$\sum_{k=1}^{\infty} |s_k t_k|$ is bounded and hence

$\sum_{k=1}^{\infty} |s_k t_k| < \delta$

(ie) $\sum_{k=1}^{\infty} s_k t_k$ converges

n approaches to '∞' in ② we get

$$\left|\sum_{n=1}^{\infty} s_n t_n\right| \leq \left|\sum_{n=1}^{\infty} s_n^2\right|^{1/2} \cdot \left(\sum_{n=1}^{\infty} t_n^2\right)^{1/2}$$

The Minkowski Inequality:

If $s = \{s_n\}_{n=1}^{\infty}$ and $t = \{t_n\}_{n=1}^{\infty}$ are in l^2, then $s+t = \{s_n + t_n\}_{n=1}^{\infty}$ is in l^2 and $\left[\sum_{n=1}^{\infty} (s_n + t_n)^2\right.$

Proof:-

by hypothesis the series $\sum_{n=1}^{\infty} s_n^2$ and $\sum_{n=1}^{\infty} t_n^2$ converges,

by the above theorem,

The series,

$\sum_{n=1}^{\infty} s_n t_n$ converges,

Since $(s_n+t_n)^2 = s_n^2 + 2 s_n t_n + t_n^2$

$\sum_{n=1}^{\infty} (s_n+t_n)^2$ converges and,

$$\sum_{n=1}^{\infty} (s_n+t_n)^2 = \sum_{n=1}^{\infty} s_n^2 + 2\sum_{n=1}^{\infty} s_n t_n + \sum_{n=1}^{\infty} t_n^2$$

using Schwartz inequality to the second term on the right, we get,

$$\sum_{n=1}^{\infty} (s_n+t_n)^2 \leq \sum_{n=1}^{\infty} s_n^2 + 2\left(\sum_{n=1}^{\infty} s_n^2\right)^{1/2}\left(\sum_{n=1}^{\infty} t_n^2\right)^{1/2} + \sum_{n=1}^{\infty} t_n^2$$

ie; $$\sum_{n=1}^{\infty} (s_n+t_n)^2 \leq \left[\left(\sum_{n=1}^{\infty} s_n^2\right)^{1/2} + \left(\sum_{n=1}^{\infty} t_n^2\right)^{1/2}\right]^2$$

Taking square root on both side

$$\left[\sum_{n=1}^{\infty} (s_n+t_n^2)\right]^{1/2} \leq \left(\sum_{n=1}^{\infty} s_n^2\right)^{1/2} + \left(\sum_{n=1}^{\infty} t_n^2\right)^{1/2}$$

Defn :-

If $s=\{s_n\}_{n=1}^{\infty}$ is an element of ℓ^2, we define $\|s\|_2$ called the norm of s, as,

$$\|s\|_2 = \left(\sum_{n=1}^{\infty} s_n^2\right)^{1/2}.$$

Properties :-

1) The norm for sequence in ℓ^2 has
$\|s_2\| \geq 0 \quad (s\in \ell^2)$

2) $\|s_2\| = 0 \Leftrightarrow s=\{0\}_{n=1}^{\infty}$

3) $\|c.s\|_2 = |c|\,\|s\|_2, \quad (c\in R, s\in \ell^2)$

4) $\|s+t\|_2 \leq \|s\|_2 + \|t\|_2, \quad (s,t\in \ell^2)$

Proof :-

1). $\|s\|_2 = \left(\sum_{n=1}^{\infty} s_n^2\right)^{1/2}$
≥ 0

2) If $\|s_2\| = 0$, then $\left(\sum_{n=1}^{\infty} s_n^2\right)^{1/2} = 0$

if and only if $\sum_{n=1}^{\infty} s_n^2 = 0$

$\Leftrightarrow \sum_{n=1}^{\infty} s_n = 0$ for all n,

ie, $s = (s_1, s_2, \dots s_n, \dots)$

$= (0, 0, \dots 0, \dots)$

$\therefore \|s\|_2 = 0$

3) $\|c\, s_2\| = |s|\, \|s\|_2$, $c \in \mathbb{R}$, $s \in l^2$

$$\|cs\|_2 = \left[\sum_{n=1}^{\infty} (cs_n)^2\right]^{1/2}$$

$$= c \left(\sum_{n=1}^{\infty} s_n^2\right)^{1/2}$$

$\therefore \|cs_2\| = |c|\, \|s\|_2$ $\quad [\because |c| = c]$

4) $\|s+t\|_2 \le \|s\|_2 + \|t\|_2$; $(s, t \in l^2)$

By Minkowski's inequality,

$$\sum_{n=1}^{\infty} (s_n + t_n)^{1/2} \le \left(\sum_{n=1}^{\infty} s_n^2\right)^{1/2} + \left(\sum_{n=1}^{\infty} t_n^2\right)^{1/2}$$

ie, $\|s+t\|_2 \le \|s\|_2 + \|t\|_2$.

Monotonic function :-

Strictly increasing function :-

The real valued function 'f' on the interval $J \subset \mathbb{R}$, is said to be strictly increasing if,

$x < y$, $x, y \in J$.

$\Rightarrow f(x) < f(y)$.

Strictly decreasing function :-

The real valued function 'f' on the interval $J \subset R$ is said to be strictly decreasing,

if $x > y$, $x, y \in J \Rightarrow F(x) > f(y)$.

Monotone function :

Monotone function is either strictly increasing or strictly decreasing.

Note :-

If 'f' is strictly increasing on J (or) strictly decreasing on J. iff f is 1-1 on J,

metric space :-

Defn :-

Let M be any set. A metric for M is a function $\rho : M \times M \to [0, \infty]$ such that,

(m_1) $\rho(x, x) = 0$, $x \in M$

(m_2) $\rho(x, y) > 0$, $(x, y \in M, x \neq y)$

(m_3) $\rho(x, y) = \rho(y, x)$, $(x, y \in M)$

(m_4) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$, $x, y, z \in M$

(Triangle Inequality)

If ρ is a metric for M then the ordered pair (M, ρ) is called a metric space.

Eg: ①

Take $M = R$ define $\rho(x,y) = |x-y|$, $x,y \in R$,

$|.|$ is a metric on R called absolute value metric (or) Euclidean metric and $(R, |\,|)$ is called Euclidean space & denote by R^1.

$M_1 : |x-x| = 0$, $x \in R$

M_2 : Let $x \neq y$, $x,y \in R$.

$\rho(x-y) = |x-y| > 0$

M_3 : $\rho(x,y) = |x-y|$

$= |(-)(y-x)|$

$= |y-x|$

$= \rho(y,x)$

$\therefore |.|$ is a metric and $(R, |.|)$ metric space,

M_4 : $\rho(x,y) = |x-y|$

$= |x-z+z-y|$

$\leq |x-z| + |z-y|$

$\leq \rho(x,z) + \rho(z,y)$

$\therefore$ equalence space is metric

Defn:-

$d : R\times R \to [0,\infty]$ by $d(x,x) = 0$, $x \in R$

$d(x,y) = 1$ $(x,y \in R, x \neq y)$

Then d is a metric on R and called discrete metric (R,d) is called discrete in space and denoted by R_d.

soln :-

$M_1 = d(x,x) = 0 \quad x \in R$

$M_2 = d(x,y) = 1 > 0 \quad [x,y \in R \ (x \neq y)]$

M_4 : let $x, y, z \in R$

(i) $x = y = z$

$0 = 0 + 0$

$d(x,y) = d(x,z) + d(z,x)$

(ii) $x = y \ , x \neq z$

(iii) $x \neq y \ , y = z$

$1 < 1 + 0$

(iv) $x \neq y \cdot x = z \quad 0 < 1 + 1$

(v) $x \neq y \neq z \ , \ 1 < 1 + 1$

$\therefore d(x,y) \leq d(x,y) + d(z,y)$

$\therefore$ d is metric

3) Fix $n \in I$, If $x = < x_1, x_2, \dots x_n >$

$y = < y_1, y_2, \dots y_n >$ are 2 ordered n tuples of real numbers, define $\rho(x,y) = \left[\sum_{k=1}^{n} |x_k - y_k|^2 \right]^{1/2}$.

Then ρ is a metric and corresponding metric space is called Euclidean n-space and denoted by E_1^n.

soln :- M_1 : $\rho(x,x)$

$= \left[\sum_{k=1}^{n} (x_k - x_k)^2 \right]^{1/2}$

$= (\Sigma 0^2)^{1/2}$

$= 0$

M_2 : $x = \langle x_1, \ldots x_n \rangle$

$y = \langle y_1 \ldots y_n \rangle$

$x \neq y \Rightarrow x_k \neq y_k$ for some $k=1$ to n.

$\Rightarrow x_k - y_k \neq 0$

$\Rightarrow (x_k - y_k)^2 \neq 0 \quad > 0$

$$\rho(x,y) = \left[\sum_{k=1}^{n} (x_k - y_k)^2\right]^{1/2} > 0$$

M_3 : $\rho(x,y)$

$$= \left[\sum_{k=1}^{n} (x_k - y_k)^2\right]^{1/2}$$

M_4 = $x = \langle x_1, \ldots x_n \rangle$, $y = \langle y_1, y_2 \ldots y_n \rangle$

$z = \langle z_1, \ldots z_n \rangle$

$$\rho(x,y) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}$$

$$= \sqrt{\left(\sum_{k=1}^{n} (x_k - z_k) + (z_k - y_k)\right]^2}$$

$$= \sqrt{\sum_{k=1}^{n} (a_k + b_k)^2}$$

$$\leq \sqrt{\sum_{k=1}^{n} a_k^2} + \sqrt{\sum_{k=1}^{n} b_k^2} \qquad [\because \text{by minkowski's Inequality}]$$

$$\leq \sqrt{\sum_{k=1}^{n} (a_k - z_k)^2} + \sqrt{\sum_{k=1}^{n} (z_k - y_k)^k}$$

$$\leq \rho(x,z) + \rho(z,y)$$

$\therefore \rho$ is metric space.

4) Let l^{∞} denote the set of all bounded sequence of real numbers, if $x = \{x_n\}_{n=1}^{\infty}$ $y = \{y_n\}_{n=1}^{\infty} \in R^n$ define $\rho(x,y) = \text{lub}\,|x_n - y_n|$ $1 \le n < \infty$,

soln :-

$$m_1: \rho(x,x) = \text{lub}_{1 \le n < \infty} |x_n - x_n|$$
$$= \text{lub}\ 0$$
$$= 0$$

$x \neq y \Rightarrow x_n \neq y_n$ for atleast one $n (n = 1,2,3,\dots)$

$$m_2: \rho(x,y) = \text{lub}_{1 \le n < \infty} (x_n - y_n)$$
$$> 0$$

$$m_3: \rho(x,y) = \text{lub}_{1 \le n < \infty} |x_n - y_n|$$
$$= \text{lub}_{1 \le n < \infty} |x_n - z_n + z_n - y_n|$$
$$\le \text{l.u.b}_{1 \le n < \infty} |x_n - z_n| + \text{l.u.b}_{1 \le n < \infty} |z_n - y_n|$$

$$\rho(x,y) \le \rho(x,z) + \rho(z,y).$$

5) consider l^2 the class of all sequences $\{s_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} s_n^2 < \infty$ for $x, y \in l^2$ define $\rho(x,y) = \|x - y\|_2$, Then ρ is a metric on l^2.

soln :-

$$(i)\ \rho(x,x) = \|x - x\|_2$$
$$= \|0\|$$
$$= 0$$

(ii) Let $x \neq y \Rightarrow x_k \neq y_k$ for atleast one the $k=1,2,\dots$

$$\rho(x,y) = \|x-y\|_2$$

$$= \sqrt{\sum_{n=1}^{\infty} (x_k - y_k)^2}$$

$\therefore \rho(x,y) > 0$

(iii) $\rho(x,y) = \|x-y\|_2$

$$= \sqrt{\sum_{n=1}^{\infty} (x_k - y_k)^2}$$

$$= \sqrt{\sum_{n=1}^{\infty} (y_k - x_k)^2}$$

$$= \|y-x\|_2$$

$\therefore \rho(x,y) = \rho(y-x)$

(iv) $\rho(x,y) = \|x-y\|_2$

$$= \|x-z+z-y\|_2$$

$$\leq \|x-z\|_2 + \|z-y\|_2$$

$\therefore \rho(x,y) \leq \rho(x,z) + \rho(z,y)$

$\therefore (\ell^2, \|\cdot\|)$ is a metric on ℓ^2.

Note :-

Consider the metric space (M, ρ) and (M_2, ρ_2).

Let $a \in M_1$, $f\ \{x \in M_1\}\ 0 < \rho_1 (r$

Limit in metric space :

Defn :-

$F(x)$ approaches L (where $L \in M_2$) as x approaches a if $\epsilon > 0$ there exists $\delta > 0$ such that

$\rho_2 (f(x), L) < \epsilon$, $0 < \rho_1(x,a) < \delta$

we write,

$\lim_{x \to a} F(x) = L$ (or) $f(x) \to L$ as $x \to a$

Def :- Subspace :-

Let $A \subset M$ and (M, ρ) is a metric space, then (A, ρ) is also a metric space called a subspace.

Eg:- [show that ρ is a metric then 2ρ is metric]

show that if ρ is a metric for a set M, then so is 2ρ, define, $d(x,y) = 2\rho(x,y) \forall x, y \in M$.

soln :-

Define $d(x,y) = 2\rho(x,y)$

(i) $d(x,x) = 2\rho(x,x) = 0$

(ii) $d(x,y) = 2\rho(x,y) \geq 0$

(iii) $d(x,y) = 2\rho(x,y)$

$= 2\rho(y,x)$

$d(x,y) = d(y,x)$

(iv) $d(x,y) = 2\rho(x,y)$

$= 2[\rho(x,z) + \rho(z,y)]$

$\leq 2.\rho[x,z] + 2\rho[z,y]$

$\therefore d(x,y) \leq \rho(x,z) + \rho(z,y)$

$\therefore$ d is metric.

(2) show that ρ and σ are two metrices for a set M then $\rho+\sigma$ is also a metric define of

$d(x,y) = (\rho+\sigma)(x,y) \Rightarrow d(x,y) + \sigma(x,y) \forall x, y \in M$.

soln :-

(i) $d(x,x) = \rho(x,x) + \sigma(x,x)$

$= \rho(0) + \sigma(0)$

$d(x,x) = 0$

(ii) $d(x,y) = \rho(x,y) + \sigma(x,y)$
> 0

(iii) $d(x,y) = \rho(x,y) + \sigma(x,y)$

$\leq \rho(x,z) + \rho(z,y) + \sigma(x,z) + \sigma(z,y)$

$d(x,y) \leq d(x,z) + d(z,y)$

$\therefore \rho + \sigma$ is metric.

Note :-

$M_1 = M_2 = R_1$, $\rho_2(F(x), L) = |F(x) - L|$

$\rho_1(x,a) = |x-a|$

Theorem :- [Property].

Let (M,ρ) be a metric space and let a be a point in M. Let f and g be real valued function whose domains are subset of M.

If $\lim_{x\to a} F(x) = L$ and $\lim_{x\to a} g(x) = N$ Then,

(i) $\lim_{x\to a} [F(x) + g(x)] = L + N$

(ii) $\lim_{x\to a} [F(x) - g(x)] = L - N$

(iii) $\lim_{x\to a} F(x)\, g(x) = LN$

(iv) If $N \neq 0$

$\lim_{x\to a} \frac{F(x)}{g(x)} = \frac{L}{N}$

Proof:-

Given : $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = N$

Given $\epsilon > 0$, There exist $\delta_1 > 0$ such that

$0 < \rho_1 (x, a) < \delta_1$

$\Rightarrow |f(x) - L| < \epsilon/2$, Then there exists $\delta_2 > 0$

such that,

$0 < \rho_2 (x, a) < \delta_2$

$\Rightarrow |g(x) - N| < \epsilon/2$

Take $\delta = \min(\delta_1, \delta_2)$

consider,

$$|f(x) + g(x) - (L+N)| = |f(x) - L) + g(x) - N)|$$
$$\leq |f(x) - L| + |g(x) - N|$$

$$|f(x) + g(x) - [L+M)| \leq \epsilon/2 + \epsilon/2$$
$$< \epsilon \qquad 0 < \rho(x, a) < \delta$$

Note : 1. Defn.

Let (M, ρ) be a metric space. If $\{s_n\}_{n=1}^{\infty}$ be a sequence of points in M. we say that $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence if given $\epsilon > 0$, There exists $N \in I$ such that $\rho(s_m, s_n) \leq \epsilon$ $(m, n \geq N)$.

Note : 2

Let (M, ρ) be a metric space and let $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence of points in M.

Then $\{s_n\}_{n=1}^{\infty}$ is a Cauchy.

Note : 3 :-

some metric space there are Cauchy sequence which are not convergent.

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UNIT-V

Continuous Functions on Metric space

Functions Continuous at a pt. on the real line;

Let $a \in R$ and f be a real valued function whose domain is some open interval $(a-h, a+h)$ (including a itself).

Defn: Continuous at a pt.

We say that the function f is continuous at $a \in R^1$ if $\lim_{x \to a} f(x) = f(a)$

Note: f is continuous at $a \in R^1 \iff$

(i). f is defined at a (or) $f(a)$ is defined

(ii). $\lim_{x \to a} f(x)$ exists

(iii). $\lim_{x \to a} f(x) = f(a)$

If any one of the above 3 conditions is not satisfy then f is not continuous at a.

Eg (1): Define f by $f(x) = \frac{\sin x}{x}$, $x \in R^1$, $x \neq 0$

(i) $f(0)$ is not defined

(ii) $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1$

(iii) $\lim_{x \to 0} f(x) \neq f(0)$

$\therefore f$ is not continuous. //

Eg (2): Define g by $g(x) = \begin{cases} \frac{\sin x}{x}, & x \in R^1, x \neq 0 \\ 0, & x = 0 \end{cases}$

(i) $g(0)$ is defined

(ii) $\lim_{x \to 0} g(x) = 1$ (iii) $\lim_{x \to 0} g(x) \neq g(0)$ $\therefore g$ is not continuous //

Eg (3). Define h by $h(x) = \begin{cases} \frac{\sin x}{x}, & x \in R^1, x \neq 0 \\ 1, & x = 0 \end{cases}$

(i), (ii) & (iii) are hold

$\lim_{x \to 0} h(x) = h(0)$, $\therefore h$ is continuous. //

Sol (a): Consider the characteristic function χ on $\mathbb{Q}$.

$$\chi_a(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in R - \mathbb{Q} \end{cases}$$, then $\chi(x)$ is defined $\forall\, x \in R'$. But $\lim_{x \to a} \chi_a(x)$ does not exists for any $a \in R'$.

$\therefore$ χ is not continuous every pt. on R.

Theorem: If the real valued function 'f' & 'g' are continuous at $a \in R'$ then so are $f+g$, $f-g$ & fg, if $g(a) \neq 0$ then f/g is also continuous at a.

Proof:

W.K.T

(i) & (ii) $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M \Rightarrow \lim_{x \to a} [f(x) + g(x)] = L + M$.

Given $\lim_{x \to a} f(x) = g(a)$, $\lim_{x \to a} g(x) = g(a)$ Consider $\lim_{x \to a} (f \pm g)(x)$

$$\lim_{x \to a} [f(x) \pm g(x)] = f(a) \pm g(a)$$
$$= (f \pm g)(a)$$

$\Rightarrow$ $f \pm g$ is continuous at a.

(iii). $\lim_{x \to a} (fg)(x) = \lim_{x \to a} f(x) \cdot g(x)$

$$= f(a) \cdot g(a) = fg\,(a)$$

(iv). $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)} = \frac{f}{g}(a)$ //

Reformulation:

Theorem: The real valued function f is continuous at $a \in R$ iff Given $\epsilon > 0$, $\exists$ $\delta > 0$ such that $|x - a| < \delta$ $\Rightarrow |f(x) - f(a)| < \epsilon$

Pf: Suppose f is continuous at $a \Leftrightarrow \lim_{x \to a} f(x) = f(a)$ iff given $\epsilon > 0$ $\exists$ a $\delta > 0$ such that $0 < |x - a| < \delta$ $\Rightarrow$ $|f(x) - f(a)| < \epsilon$. If $x = a$ then also $|x - a| < \delta$ and $|f(x) - f(a)| < \epsilon \Leftrightarrow$ Given $\epsilon > 0$ $\exists$ a $\delta > 0$ s.t $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$ //

Defn: Open ball

If $a \in R^1$ and $r > 0$, we define $B[a,r]$.

$B[a,r] = \{x \in R^1 : |x-a| < r\}$ and call it as an open ball of radius 'r' about a.

Note: $B[a:r] = (a-r, a+r)$ open interval in R^1.

Theorem: The real valued function 'f' is contin/.. at $a \in R^1$, iff the inverse image under f of open ball $B[f(a):\epsilon]$ about $f(a)$ contains an open ball $B[a, \delta]$ about a.

(ie) Given $\epsilon > 0$ $\exists$ $\delta > 0$ such that $B[a,\delta] \subset f^{-1} B[f(a), \epsilon]$

(or)

$f^{-1} B[f(a), \epsilon] \supset B[a, \delta]$

Pf: Suppose that f is continuous at $a \in R^1$

$\Leftrightarrow \lim_{x \to a} f(x) = f(a)$

$\Leftrightarrow$ Given $\epsilon > 0$ $\exists$ a $\delta > 0$ such that

$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

$\Leftrightarrow x \in B[a, \delta] \Rightarrow f(x) \in B[f(a):\epsilon]$

$\Leftrightarrow x \in B[a, \delta] \Rightarrow x \in f^{-1}[B[f(a):\epsilon]]$

$\Leftrightarrow B[a, \delta] \subset f^{-1}[B[f(a);\epsilon])$

Theorem:

(X) The real valued functions f is continuous at $a \in R^1$ iff whenever $\{x_n\}_{n=1}^{\infty}$ is a seq. of real nos. cgs. to 'a', then the seq. $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(a)$. (ie) f is continuous at 'a' iff $\lim_{n \to \infty} x_n = a \Rightarrow \lim_{n \to \infty} f(x_n) = f(a)$.

Proof: Let f be continuous at $a \in R^1$, Given $\epsilon > 0$ $\exists$ $\delta > 0$ such that $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$ —①

Let $\lim_{n \to \infty} x_n = a \Rightarrow$ for $\delta > 0$ $\exists$ $N \in I$ s.t $|x_n - a| < \delta$ $(n \geq N)$

from ① $|x_n - a| < \delta \Rightarrow |f(x_n) - f(a)| < \epsilon$ $(n \geq N)$

$\lim_{n \to \infty} f(x_n) = f(a)$

Conversely, suppose that $\lim_{n\to\infty} x_n = a \Rightarrow \lim_{n\to\infty} f(x_n) = f(a)$.

To prove: f is Conty at $a \in R^1$, on the Contrary assume that f is not Conty at 'a'. (ie) for some $\epsilon > 0$ & for every $\delta > 0$

$$|x-a| < \delta \quad \text{But} \quad |f(x) - f(a)| \geq \epsilon$$

Take $\delta = \frac{1}{n}$, $n \in I$ then $\exists$ a seq. $\{x_n\}_{n=1}^{\infty}$ s.t

$$|x_n - a| < \frac{1}{n}, \quad \text{But} \quad |f(x_n) - f(a)| \geq \epsilon, \quad n = 1, 2 \ldots$$

$\Rightarrow \lim_{n\to\infty} x_n = a$, But $\lim_{n\to\infty} f(x_n) \neq f(a)$.

which is Contradicts, f must be Continuous at a.

Theorem:

If f and g are r.v.f Conty. at 'a' and $f(a)$ respectively then $g \circ f$ is also Conty. at 'a'.

Proof: Given $\lim_{x\to a} f(x) = f(a)$, $\lim_{y \to b} g(y) = g(b)$, where $y = f(x)$, $b = f(a)$

f is Conty. at 'a' and g is Conty. at $f(a)$.

let $\lim_{n\to\infty} x_n = a$, since f is Conty. at a, we have

$$\lim_{n\to\infty} f(x_n) = f(a)$$

since g is Conty. at $f(a)$, we has

$$\lim_{n\to\infty} g[f(x_n)] = g[f(a)]$$

$$\lim_{n\to\infty} (g\circ f)(x_n) = (g \circ f)(a)$$

(ie) $\lim_{n\to\infty} x_n = a \Rightarrow \lim_{n\to\infty} (g\circ f)(x_n) = (g\circ f)(a)$

$\therefore$ $g \circ f$ is Conty. at a.

Functions Continuous on a Metric space:

Open ball: Let $\langle M, \rho \rangle$ be a metric space. If $a \in M$ and $r > 0$ then we define

$$B[a; r] = \{ x \in M : \rho(x, a) < r \}$$

we call $B[a, r]$ the open ball of radius r.

Eg : (1) In R^3, $B[0,1] = \{(x,y,z) \in R^3,\ x^2+y^2+z^2<1\}$ open unit ball.

In R^2, $B[0,1] = \{(x,y) \in R^2,\ x^2+y^2<1\}$ open unit circular disc.

In R^1, $B[0,1] = \{x \in R^1,\ |x|<1$
$= (-1,1)$ open interval.

Eg (2) : Take $M = [0,1]$, the closed interval with absolute value metric.

Eg (3) : $M = Rd$
$B[a,1] = \{x \in Rd;\ d(x,a)<1\} = \{a\}$
$B[a,2] = \{x \in Rd;\ d(x,a) < 2\} = Rd$

Defn : Let (M_1,ρ_1) (M_2,ρ_2) be the two metric space. Let $a \in M_1$, Let 'f' be any fun. with domain as some open ball $B[a,h]$, $h>0$ and its range as a subset of M_2.

Defn : The function 'f' is cont. at $a \in M_1$, $\lim_{x\to a} f(x) = f(a)$

Theorem (1) Properties of limit in metric space :

The limit of a product is equal to the products of the limits.
(or)
$$\lim_{x\to a} [f(x) \cdot g(x)] = LN.$$

proof : If $\lim_{x\to a} f(x) = L$ & $\lim_{x\to a} g(x) = N$

We have to prove that
$$\lim_{x\to a} f(x)\cdot g(x) = LN$$
$$\rightarrow |f(x)\, g(x) - LN| < \epsilon$$

Consider

$$|f(x)\cdot g(x) - LN| = |f(x)g(x) - Lg(x) + Lg(x) - LN|, \text{ whenever } 0<|x-a|<\delta$$
$$\leq |f(x)g(x) - Lg(x)| + |g(x) - LN|$$
$$\leq |g(x)|\,|f(x) - L| + |L|\,|g(x) - N|$$

since $g(x)$ is bounded. $0<|x-a|<\delta$.

$|g(x)| < K$ for all values of x such that

Thus $|f(x)g(x) - LN| < K\,|f(x) - L| + |L|\,|g(x) - N|$

Any $\epsilon > 0$, we can find a +ve no. $\delta < \delta'$ s.t $|f(x) - L| < \frac{\epsilon}{2K}$, $0<(x-a)<\delta$

and $|g(x) - N| < \dfrac{\epsilon}{2|L|+1}$

Thus $|f(x)g(x) - LN| < K\cdot\frac{\epsilon}{2K} - |L|\frac{\epsilon}{2|L|+1}$

$< \epsilon$, whenever $0<|x-a|<\delta$

$\therefore \lim\limits_{x\to a} f(x)\cdot g(x) = LN$ //

Thm(2) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = L/N$, $N \neq 0$

Pf: Given $\lim\limits_{x\to a} f(x) = L$, $\lim\limits_{x\to a} g(x) = N$,

consider $\left|\dfrac{f(x)}{g(x)} - \dfrac{L}{N}\right| = \left|\dfrac{f(x)}{g(x)} - \dfrac{f(x)}{N} + \dfrac{f(x)}{N} - \dfrac{L}{N}\right|$

$$= \left|\frac{f(x)}{Ng(x)}(N - g(x)) + \frac{1}{N}(f(x) - L)\right|$$
$$\leq \frac{|f(x)|}{|N||g(x)|}|N - g(x)| + \frac{1}{|N|}|f(x) - L| \quad \text{— (1)}$$

The fun, f & g are bdd. Let l be the upper bound of $|f|$ and M be lower bound for $|g|$ so that $|f(x)| < l$ & $|g(x)| > M$.

(1) $\Rightarrow \left|\dfrac{f(x)}{g(x)} - \dfrac{L}{N}\right| \leq \dfrac{l}{M|N|}|N - g(x)| + \dfrac{1}{|N|}|f(x) - L|$, Given $\epsilon > 0$,

we find +ve nos. δ_1 & δ_2 s.t $|f(x) - L| < \frac{\epsilon}{2}|N|$ $0<|x-a|<\delta_1$,

$|g(x) - N| < \dfrac{M|N|}{l}\cdot\dfrac{\epsilon}{2}$, $0<|x-a|<\delta_2$

Choose $\delta = \min(\delta_1, \delta_2)$

$$\left|\frac{f(x)}{g(x)} - \frac{L}{N}\right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$< \epsilon, \quad 0<|x-a|<\delta$$

$\therefore \lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = L/N$ //

Open sets.

Defn: Let M be metric space, we say that the subset G of M is an open subset of M. (or G is open). If for every $x \in G$ $\exists$ $r>0$ such that $B[x,r] \subset G$.

Result: Every open ball in a Metric space is an open set.

Pf: Let M be a M.S, $a \in M$. Consider the open ball

$B = B[a, \delta]$

To Prove: B is an open set. Let $x \in B$ and let $\rho[a,x] = r$

Take $t < \delta - r$ $t > 0$, Consider the open ball $B[x,t]$

Claim: $B[x,t] \subset B$, let $y \in B[x,t] \Rightarrow \rho(x,y) < t$

Consider $\rho(y,a) \leq \rho(y,x) + \rho(x,a)$

$$< t + r$$
$$< \delta - r + r$$
$$< \delta .$$

$y \in B$, $\because y$ is an arbitrary.

$B[x,t] \subset B$, $\therefore$ B is open.

Eg:

(1). In R^2 set of all pts ~~~~ inside an ellipse is an open set.

(2). In R^2 " " " " an circle is an open set.

(3). In R^3 " " " a sphere " " "

(4) In Rd, If $a \in Rd$, then $\{a\} = B[a,1]$

$\therefore$ $\{a\}$ is open set in Rd.

(ii). Every singleton set in Rd is not open.

Note: (5). In R^1 set containing an element is not open.

$[0, \frac{1}{2})$ is not open in R^1, But in the m.s $[0,1]$ is open.

Theorem: Let $\mathcal{F}$ be any non-empty family of open subsets of m.s. M. Then $\bigcup_{G \in \mathcal{F}} G$ is also an open subset of M.

(or)

P.T union of arbitrary collection of open sets is an open set.

Pf: Let $H = \bigcup_{G \in \mathcal{F}} G$. If $H = \phi$, then H is open.
If $H \neq \phi$ then $x \in H \Rightarrow \bigcup_{G \in \mathcal{F}} G$

$\Rightarrow x \in G$, for atleast one $G \in \mathcal{F}$

$\Rightarrow$ G is open so $\exists$ some open ball $B[x, r) \subset G$

$\Rightarrow B[x, r] \subset \bigcup G = H$, since x is arbitrary

$\Rightarrow$ H is open.

Theorem: Every subset of R_d is open.

Pf: Let A be any subset in R_d.

$A = \bigcup_{a \in A} \{a\}$ and $\{a\}$ is open.

Since arbitrary union of open sets is open.

$\therefore$ A is open.

Note: Arbitrary infinite intersection of open sets need not be open.

Eg: In R^1, let $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$, $n \in I$

Each I_n is open.

$\bigcap_{n=1}^{\infty} I_n = \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ is not open in R^1.

$\Rightarrow$

Theorem: If G_1 and G_2 are open subset of the m.s M. Then $G_1 \cap G_2$ is open in M.

Pf: Let $H = G_1 \cap G_2$, If $H = \varphi$, then H is open.

If $H \neq \varphi$ then $\exists\ x \in H$

$\Rightarrow x \in G_1$ and $x \in G_2$ $\exists\ r_1 > 0$ & $r_2 > 0$

such that $B[x, r_1] \subset G_1$ and $B[x, r_2] \subset G_2$ [$\because G_1$ & G_2 are open].

[Take $r = \min[r_1, r_2]$ Then $B[x, r] \subset G_1$ & $B[x, r] \subset G_2$

$\Rightarrow B[x, r] \subset G_1 \cap G_2$

$\Rightarrow G_1 \cap G_2$ is open.

Result: The intersection of any finite no. of open set is open.

Proof: By induction, Let $P(n) = \bigcap_{k=1}^{n} G_k = G_1 \cap G_2 \cdots \cap G_n$ when G_k is also open.

case (i), $n = 2$

$P(2) = G_1 \cap G_2$, G_1 & G_2 are open.

(By above theorem) $\Rightarrow G_1 \cap G_2$ are open.

Case (ii) Assume that $p(m)$ is open for $n = m$.

(ie) $p(m) = \bigcap_{k=1}^{m} G_k$ is open.

$$P(m+1) = \bigcap_{k=1}^{m+1} G_k = \bigcap_{k=1}^{n} G_k \cap G_{k+1}$$

Since $\bigcap_{k=1}^{m} G_k$ & G_{m+1} are open.

$\Rightarrow \bigcap_{k=1}^{m} G_k \cap G_{m+1}$ is open by case (i).

$\Rightarrow \bigcap_{k=1}^{m} G_k$ open. By induction hypothesis the result is true for any finite $n \in N$.

Theorem: Every open subset G of R^1 can be written $G = \cup I_n$, $n \in I$, $I_1 \ldots$ are finite no/. (or)

Countable no of open intervals which are mutually disjoint (i.e) $I_m \cap I_n = \phi$ if $m \neq n$.

Pf: If $x \in G$, then there is an open interval open ball B containing x s.t $B \subset G$.

Let I_x denote the largest open interval containing x such that $I_x \subset G$. Then $G = \bigcup_{x \in G} I_x$, Now if $x, y \in G$ then either $I_x = I_y$ (or) $I_x \cap I_y = \phi$, for if $I_x \neq I_y$ and $I_x \cap I_x \neq \phi$ then $I_x \cup I_y$ is an open interval contained in G which is large than I_x. This contradicts to the definition of I_n.

To prove: I_x are countable.

Each I_x contains a rational no/. since disjoint intervals can not contains the same rational no/. and since there are only countably many rationals, then can be finite (or) countable no/. of mutually disjoint interval I_x.

Theorem: Let (M_1, ρ_1) & (M_2, ρ_2) be metric space and let $f: M_1 \to M_2$ then f is continuous on M iff $f^{-1}(G)$ is open in M_1 whenever G is open in M_2 (ie) f is continuous if and only if the inverse image of open set is open.

Proof: suppose f is continuous, let G be an open in M_2.

To Prove: $f^{-1}(G)$ is open in M. Let $x \in f^{-1}(a)$

$\Rightarrow f(a) \in G$

since G is open, $\exists$ an open ball $B[f(x), \epsilon] \subset G$.

since f is continuous at every x. $\exists$ an open ball

$$B[x, \delta] \subset f^{-1}\left(B[f(x), \epsilon]\right)$$

$$\subset f^{-1}(G)$$

$x \in B[x, \delta] \subset f^{-1}(G)$, $f^{-1}(G)$ is open.

Conversely assume that whenever G is open in M_2.

$\Rightarrow f^{-1}(G)$ is open in M.

To Prove: f is continuous on M.

(ie) It is sufficient to prove that f is continuous at an arbitrary point $a \in M$.

Let B: $B[f(a), \epsilon] \subset M_2$, B is open in M_2.

By assumption $f^{-1}(B)$ is open in M_1.

$a \in f^{-1}(B)$ & $f^{-1}(B)$ is open $\exists$ an open ball

$B[a, \delta] \subset f^{-1}[B]$. $\therefore$ f is continuous at a.

Closed set

Defn: Let E be a subset of metric space M. A point $x \in M$ is called a limit point of E, if $\exists$ a sequence $\{x_n\}$ of points in E which converges to x.

The set $\bar{E}$ of all its points of E is called the closure of E.

Cor: If E is any subset of the metric space M then $E \subset \bar{E}$.

Pf: Let $x \in E$, then sequence $x_1, x_2 \cdots$ of points in E converges to x.

$\therefore$ x is a limit point of E.

$\Rightarrow x \in \bar{E}$, since x is arbitrary.

$E \subset \bar{E}$.

Note: E need not be equal to $\bar{E}$

Eg: $E = (0, 1] \subset R^1$, then 0 is the limit pt. of E. Since If a sequence $\{\frac{1}{n}\}$ of points in E converges to 0. But $0 \notin E$,

$\therefore \bar{E} \not\subset E$

$\therefore \bar{E} \neq E$

Result: Let E be the subset of Metric space M, we say that E is closed subset of M, if $E = \bar{E}$, always $E \subset \bar{E}$.

To prove E is closed, It is sufficient to prove that $\bar{E}$ contained in E.

Theorem: Let E be a subset of the metric space M. Then the pt. $x \in M$ is a limit point of E iff every open ball $B[x; r]$ about x contains atleast one point of E.

Proof: Let x be a limit point of E. Then there exists sequence $\{x_n\}_{n=1}^{\infty}$ of points in E converges to x. Given $r > 0$ then there exists $N \in I$ s.t. $\rho(x_n, x) < r$, $n \geq N$.

$x_n \in B[x; r]$, $n \geq N$ $x_n \in E$

$\therefore$ Every open $B[x; r]$ contains atleast one point of E.

Conversely,

suppose that every open ball about x contains atleast one point of E. Let $x_n \in B[x, \frac{1}{n}]$, $x_n \in E$ $(n=1,2 \dots)$

$$\Rightarrow \rho(x_n, x) < \frac{1}{n} \quad (n=1,2 \dots)$$

$$\Rightarrow \rho(x_n, x) \to 0 \text{ as } n \to \infty$$

$\lim_{n \to \infty} x_n = x$ $\therefore$ x is the limit point of E.

Eg: If L is a st. line in $\mathbb{R}^2$ then no points outside of L can be a limit points of L. (ie) L contains all its limit point.

$\therefore$ L is closed in $\mathbb{R}^2$.

Result: In Rd every singleton set is open and closed.

Solution: Let $A = \{x\}$ every $x \in A$, $B[x, 1] = \{x\} \subset A$

$\therefore$ A is open. Also in any metric space singleton set closed. $\therefore$ A is closed. A is both open and closed.

Theorem: If E is any subset of a metric space M, then $\overline{E}$ is closed (ie) $\overline{E} = \overline{\overline{E}}$.

Proof: Always $\overline{E} \subset \overline{\overline{E}}$ — (1)

To prove: $\overline{\overline{E}} \subset \overline{E}$, let $x \in \overline{\overline{E}}$, to prove $x \in \overline{E}$

It is enough to prove that any open ball $B[x; r]$ contains atleast one point of E. since $x \subset \overline{E}$ (x is the limit pt. of E)

$\Rightarrow$ every open ball $B[x; r]$ contains atleast one pt. of $y \in \overline{E}$. Let $\rho(x,y) = s$ take $t < r - s$

Consider the open ball $B[y, t]$, $y \in E \Rightarrow$ open ball $B[y, t]$ contains atleast one point $z \in E$, $\rho(y, z) < t$

Consider
$$\rho(x, z) \le \rho(x,y) + \rho(y,z)$$
$$< s + t$$
$$< s + r - s < r \Rightarrow z \in B[x; r]$$

$B[x';r]$ Contains atleast one pt. $x \in E$, x is a limit pt. of E.

$\Rightarrow x \in \bar{E}$

since x is a arbitrary $\bar{\bar{E}} \subset \bar{E}$ — (2)

from (1) & (2) $\Rightarrow \bar{E} = \bar{\bar{E}}$

$\therefore$ E is closed.

Theorem:

In any metric space (M, ρ) the set M and φ are both closed.

solution: since M contains all its limit pt. so M is closed.

φ has no limit pt. It contains all its limit points.

$\therefore$ M and φ are both closed.

Theorem: If F_1 and F_2 are closed subsets of a metric space M, then $F_1 \cup F_2$ is also closed.

Proof: Let F_1 and F_2 are closed subset of M.

$\Rightarrow F_1 = \bar{F_1}$ and $F_2 = \bar{F_2}$. Let $H = F_1 \cup F_2$

If $H = \varphi$ then there is nothing to prove.

If $H \neq \varphi$ always $F_1 \cup F_2 \subset \overline{F_1 \cup F_2}$ — (1)

To Prove $\overline{F_1 \cup F_2} \subset F_1 \cup F_2$, let $x \in \overline{F_1 \cup F_2}$

$\Rightarrow$ x is a limiting point of $F_1 \cup F_2$.

$\Rightarrow$ Every open ball $B[x;r]$ Contains atleast one point of $F_1 \cup F_2$ (ie) Every open ball $B[x,r]$ contains atleast one pt. of F_1 (or) one pt. of F_2. x is pt. of F_1 or F_2

$\Rightarrow x \in \bar{F_1} = F_1$ (or) $x \in \bar{F_2} = F_2$.

$\Rightarrow x \in F_1 \cup F_2$, $\therefore \overline{F_1 \cup F_2} \subset F_1 \cup F_2$ — (2)

From (1) & (2)

$F_1 \cup F_2 = \overline{F_1 \cup F_2}$

$\therefore F_1 \cup F_2$ is closed.

Result:

If $C_1, C_2 \ldots C_n$ are closed subset of a metric space (x,d) then $C_1 \cup C_2 \cup \ldots \cup C_n$ is also closed subset of S.

Note:

The union of infinite no. of closed set need not be closed.

Theorem: If $\mathcal{F}$ closed subset of metric space M, then $\bigcap F$ is closed $F \in \mathcal{F}$ (or)

An arbitrary intersection of closed subset of a metric M is also closed.

Proof: $H = \bigcap F$, Each $F \in \mathcal{F}$, F is closed.

If $H = \varphi$ then there is nothing to prove.

Let $H \neq \varphi \Rightarrow F = \overline{F} \quad \forall F \in \mathcal{F}$, always

$$\bigcap_{F \in \mathcal{F}} F \subset \bigcap_{F \in \mathcal{F}} \overline{F} \quad \text{—(1)}$$

To prove: $\bigcap_{F \in \mathcal{F}} \overline{F} \subset \bigcap_{F \in \mathcal{F}} F$, let $x \in \bigcap_{F \in \mathcal{F}} \overline{F}$

$\Rightarrow x$ is a limit pt. of $\bigcap_{F \in \mathcal{F}} F$. Every open ball $B[x, r]$ contains atleast one pt. of $\bigcap_{F \in \mathcal{F}} F$

$\Rightarrow$ Every open ball $B[x; r]$ contains atleast one pt. of F, $\forall F \in \mathcal{F}$

x is a limit pt. of Every F.

$\Rightarrow x \in \overline{F} = F \quad (\forall F \in \mathcal{F})$

$\Rightarrow x \in \bigcap_{F \in \mathcal{F}} F$, $\therefore \bigcap_{F \in \mathcal{F}} \overline{F} \subset \bigcap_{F \in \mathcal{F}} F$ —(2)

From (1) & (2)

$$\bigcap_{F \in \mathcal{F}} F = \bigcap_{F \in \mathcal{F}} \overline{F}$$

$\therefore \bigcap_{F \in \mathcal{F}} F$ is closed //

Theorem:
Let G be an open subset of the metric space M then $G' = M - G$ is closed conversely if F is closed subset of M then $F' = M - F$ is open.

(or)

Complement of an open set is closed conversely Complement of closed set is open.

Proof: Let G be an open subset of a metric space M.

To prove: $G' = M - G$ is closed.

Let $x \in G$, since G is open, then their exists an open $B[x, r] \subset G$, $B[x, r] \cap G' = \phi$

(ie) No point of G' is in $B[x, r]$, x is not a limit pt. of G'. G' contains all its limiting points. G' is closed.

Conversely, suppose that F is closed in M.

To prove: $F' = M - F$ is open

Let $x \in F' \Rightarrow x \notin \bar{F} = F$, x is not a limiting pt. of F, then there exists an open ball $B[x, r]$ which contains no point of F.

$\Rightarrow B[x, r] \subset F'$, $\therefore F'$ is open.

Theorem:
Let (M_1, ρ_1) & (M_2, ρ_2) be a metric space and let $f: M_1 \to M_2$ then f is continuous on M_1 iff $f^{-1}(F)$ is a closed subset of M_1 whenever F is a closed subset of M_2.

(or)

F is continuous on a metric space iff inverse image of a closed set is closed.

Proof:

Suppose F is Continuous on M_1. Let $F \subset M_2$ is a

To prove: $f^{-1}(G)$ is closed. F is closed on M_2.

$\Rightarrow$ F^{-1} is open in M_2. Since F is Continuous $f^{-1}(G)$ is open in M_1. But $F \cup F' = M_2 \Rightarrow f^{-1}(F) = f^{-1}(M_2) = M_1$

since $f^{-1}(F')$ is open in M_1, its Complement $f^{-1}(F)$ is closed in M.

Conversely, Assume that F is closed subset of M_2.

$\Rightarrow f^{-1}(F)$ is closed in M_1.

To prove, f is continuous on M_1. Let G be an open set of M_2. G' is closed in M_2. $f^{-1}(G)$ is closed in M.

But $G_1 \cup G' = M_2$. $f^{-1}(G) \cup f^{-1}(G) = f^{-1}(M_2) = M_1$, Its Complement

$f^{-1}(G)$ is open in M_1. $\therefore$ F is Continuous on M.

Defn: If $f: M_1 \rightarrow M_2$ is 1-1 and onto. f, and f^{-1} are Continuous then f is said to be homomorphism.

Defn: Dense subset

Let M be a metric space. The subset A of M is said to be dense in M if $\bar{A} = M$

(i.e) A is dense in M if every point in M is a limit pt. of A.

Eg: The A of rational is dense in R^1.

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Seen.

Reneta 26/7/2024